INTRODUCTION TO BEAM DYNAMICS

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Part One – Introduction

Let's look at a synchrotron

Periodicity and stability

Hierarchy of beam descriptions

SHM and the pendulum

Newton and Hamilton

Part Two - Hill's equations

Beams and magnets

Derivation of Hill's equations

The transfer matrix approach

Matrix properties

Part Three – Lattice functions

The Courant-Synder formalism

How to transform the lattice functions

Stability

Tune

Part Four – Optics and lattice design

The FODO cell

Lattice design

Mini beta insertions

Principles of lattice design

Optical structures

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Part Five- Errors in our lattices

Field errors

Closed orbit distortion

Tune shifts

Part Six – Real particles

Dispersion

Momentum compaction

Chromaticity

Emittance

Part Seven– Longitudinal dynamics

The principle of phase stability

The pill box cavity and real cavities

Longitudinal dynamics

Buckets and bunches

Part Eight– Radiation effects

Synchrotron radiation

Damping effects (directed reading)

Learning material for this course

Course Material

CI Courses (
http://www.cockcroft.ac.uk/pages/education.htm)

Linear Dynamics (Andy Wolski)

Accelerator Physics (Rob Appleby)

Linear Optics and Lattice Design (Bernhard Holzer)

Optics codes (Hywel Owen)

RF (Roger Jones, Graeme Burt)

Relativity (Chris Prior)

Electromagnetism

And so ....

CAS (http://www.cern.ch/CAS)

And many more....

Books

Lee, "Accelerator physics"

Wiedemann, "Particle accelerator physics"

Chao and Tigner, "The handbook"

Wille, "The physics of particle accelerators"

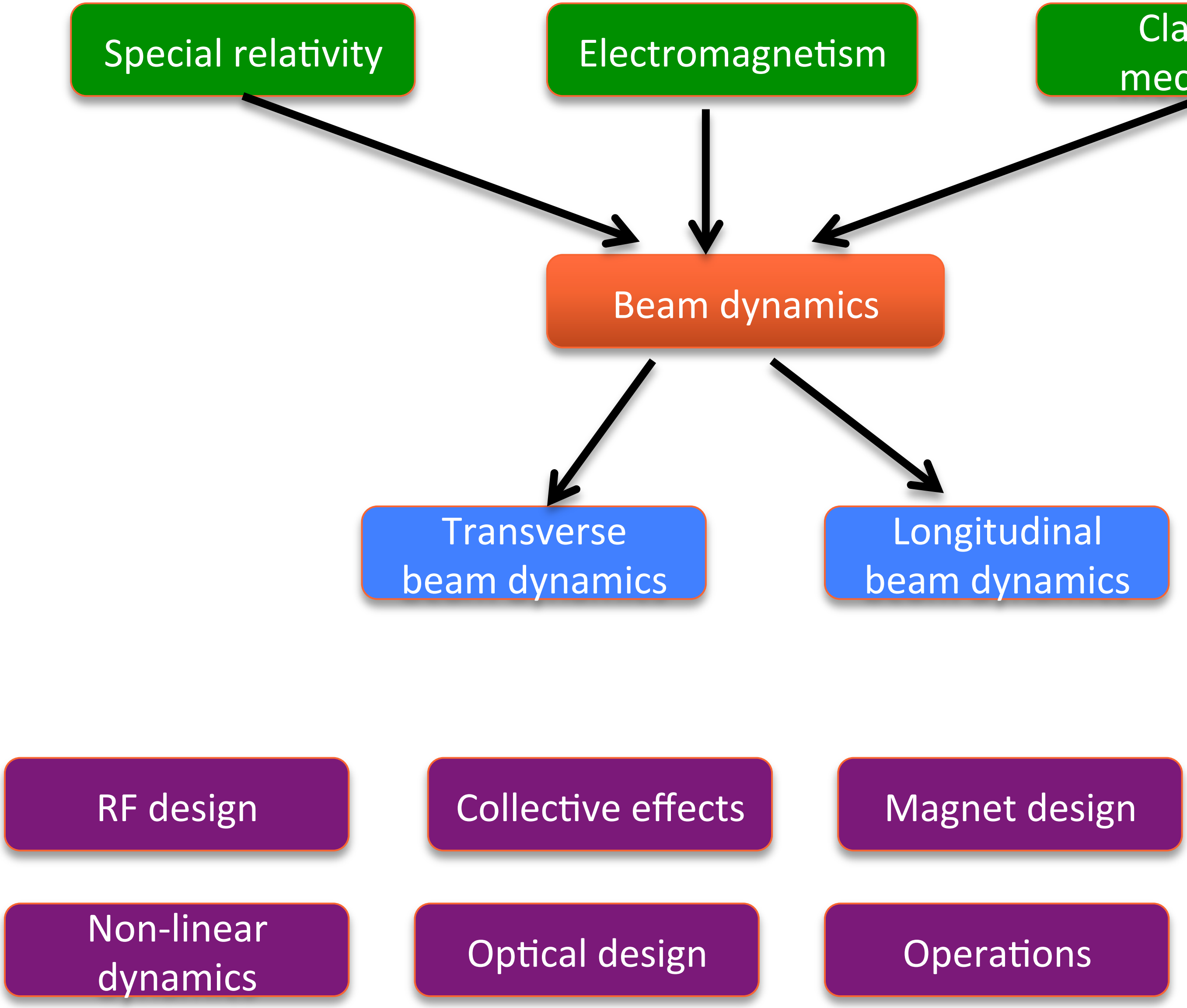
Forest, "Beam dynamics"

Wangler, "RF linear accelerators"

And many more....

(available from all good booksellers!)
(and amazon)

The pathway of an accelerator physicist



+ many others

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PART ONE - INTRODUCTION

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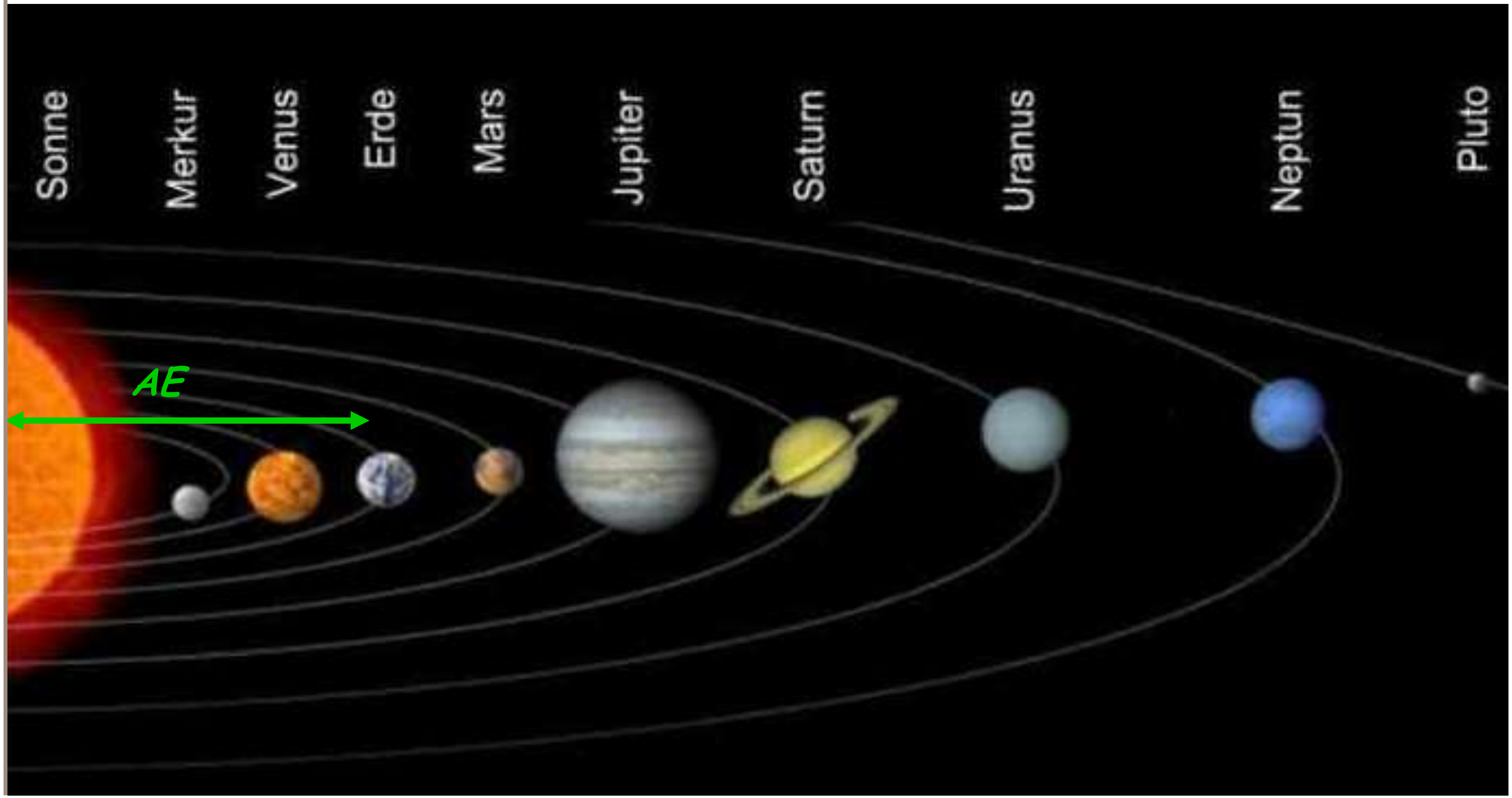
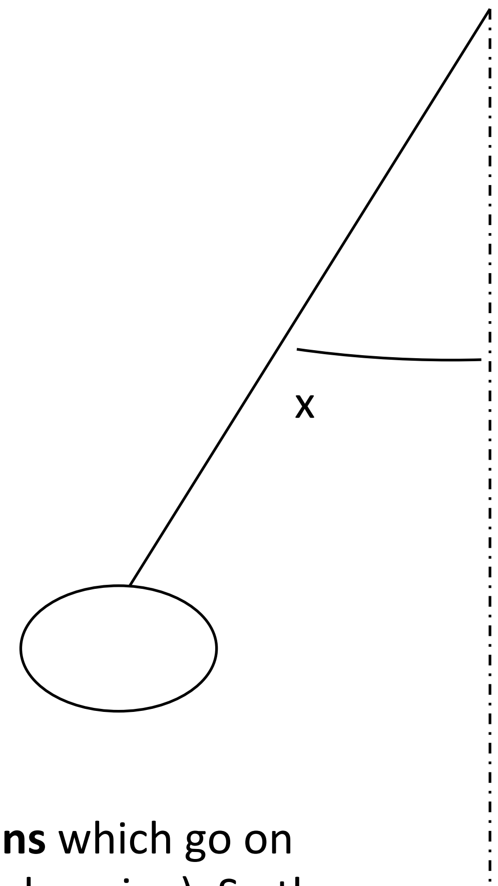
Largest storage ring: The Solar System

astronomical unit: average distance earth-sun
$1AE \approx 150 * 10^6$ km
Distance Pluto-Sun ≈ 40 AE



A pendulum and a synchrotron

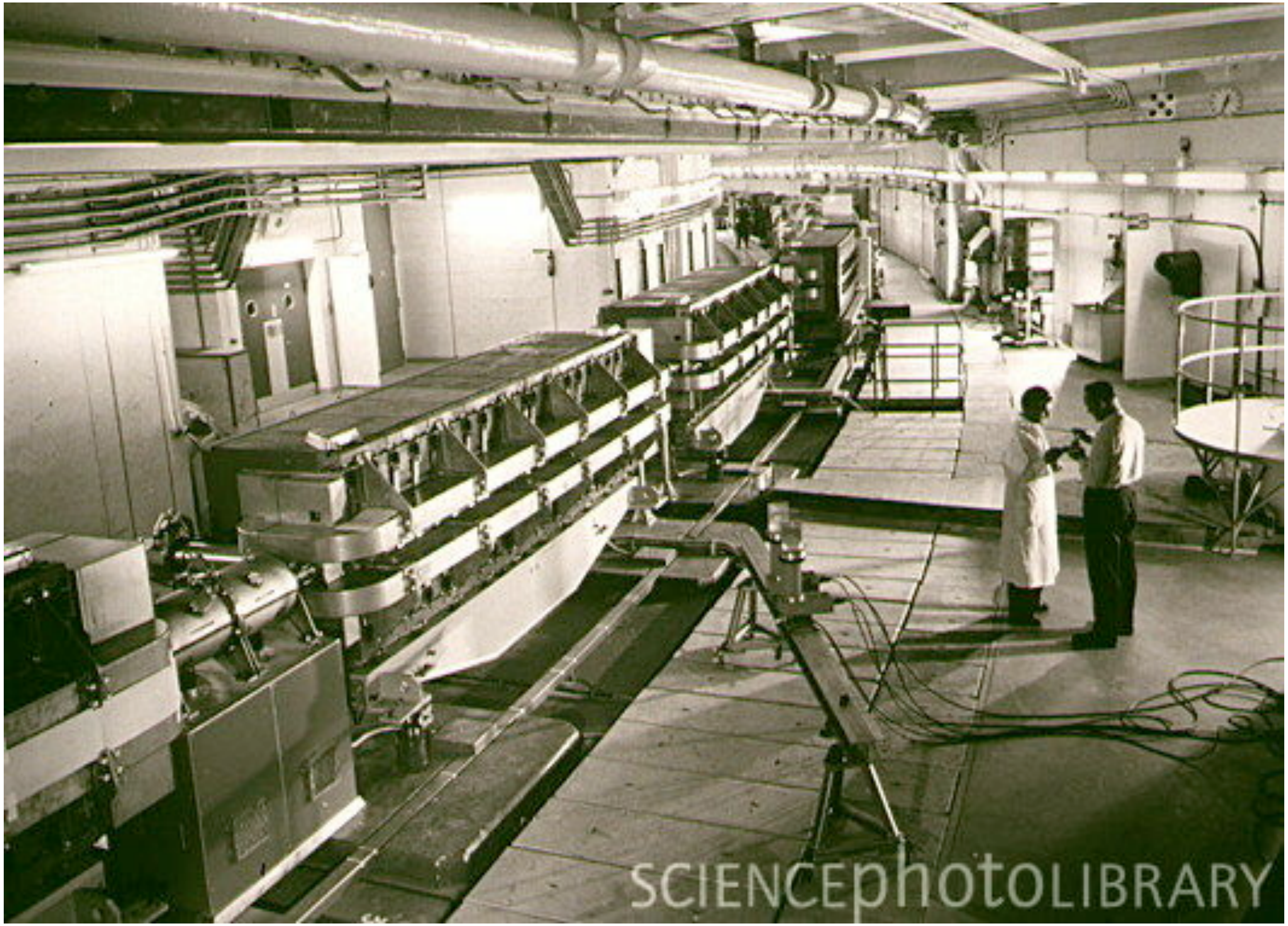
Let's look at a pendulum….



We see **periodic oscillations** which go on forever (in the absence of damping). So the motion is stable

The period depends on gravity and the length of the pendulum.

A synchrotron (PS)



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DIAMOND light source

An alien looks…

Now imagine you are an alien and come to earth to look at the PS…

What observations do you make and what would impress you?

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An alien looks…

1) The particles spend a long time in the ring. The motion is STABLE

2) The machine is periodic

3) The particle motion does not have the same periodicity as the machine on a particle by particle basis but the envelope of the motion follows the machine periodicity.

These are observations we will study and explain using beam dynamics

I would add one more, which will become clearer later

4) The motion is symplectic in hadron machines and nearly symplectic in electron machines (but also slightly stochastic)

The LHC

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LHC parameters

Nominal LHC parameters	
Beam injection energy (TeV)	0.45
Beam energy (TeV)	7.0
Number of particles per bunch	1.15×10^{11}
Number of bunches per beam	2808
Max stored beam energy (MJ)	362
Norm transverse emittance (μm rad)	3.75
Colliding beam size (μm)	16
Bunch length at 7 TeV (cm)	7.55

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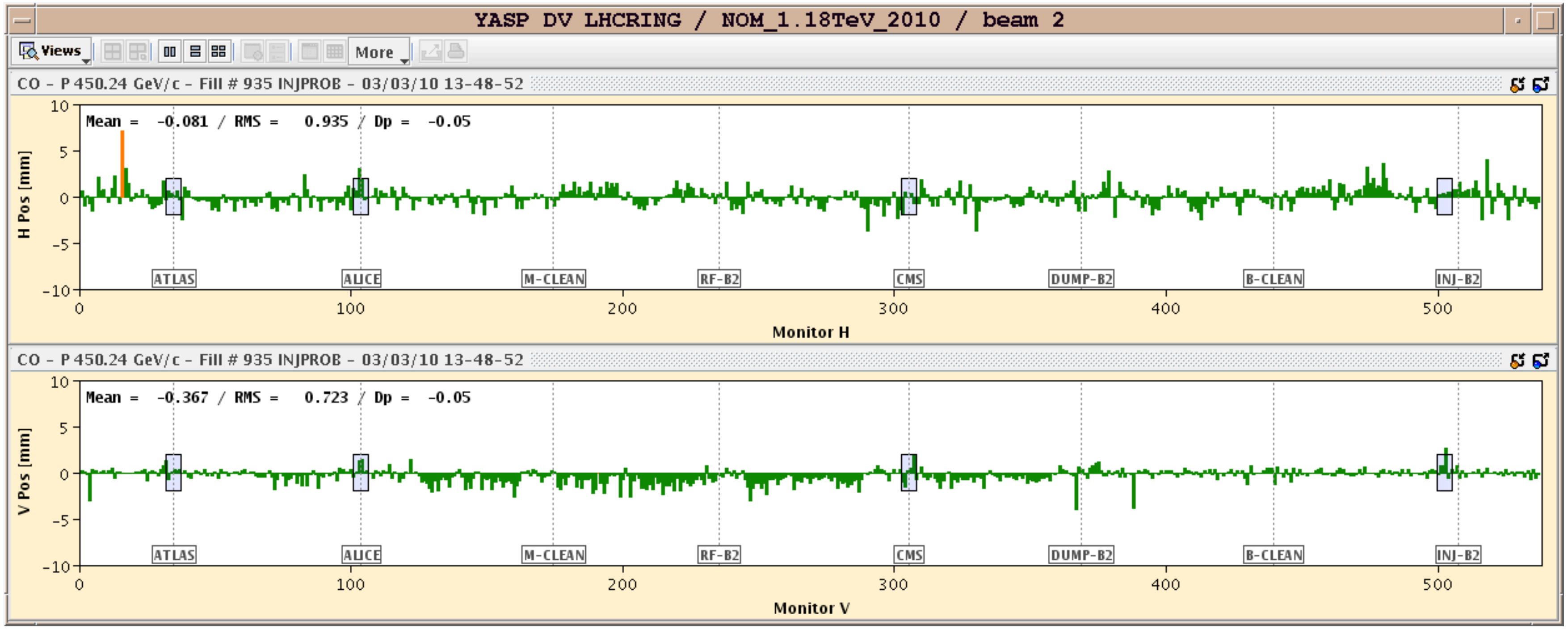
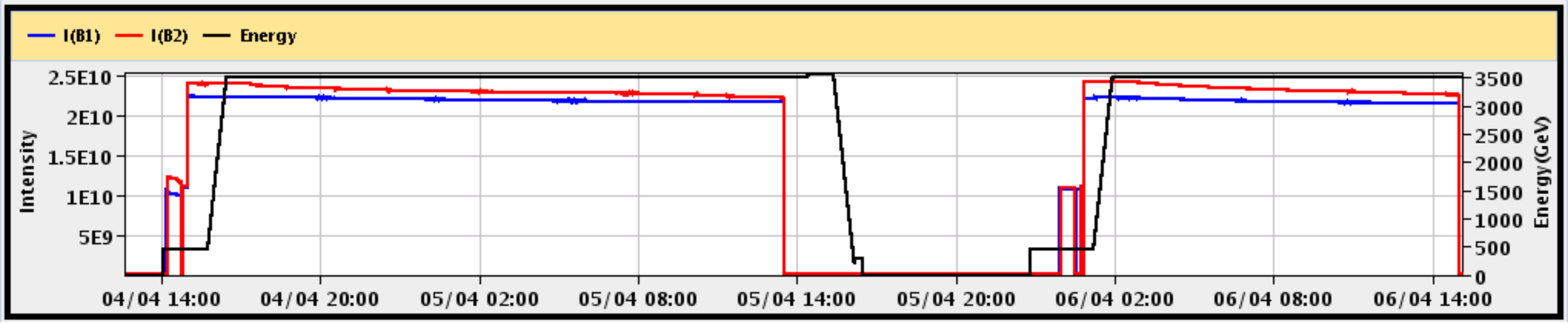
The closed orbit of the LHC



There is a closed design orbit around the ring, and all particles oscillate around this design orbit by a small amount (~ mm)

We need to focus the beam around this closed orbit

The lifetime of the LHC



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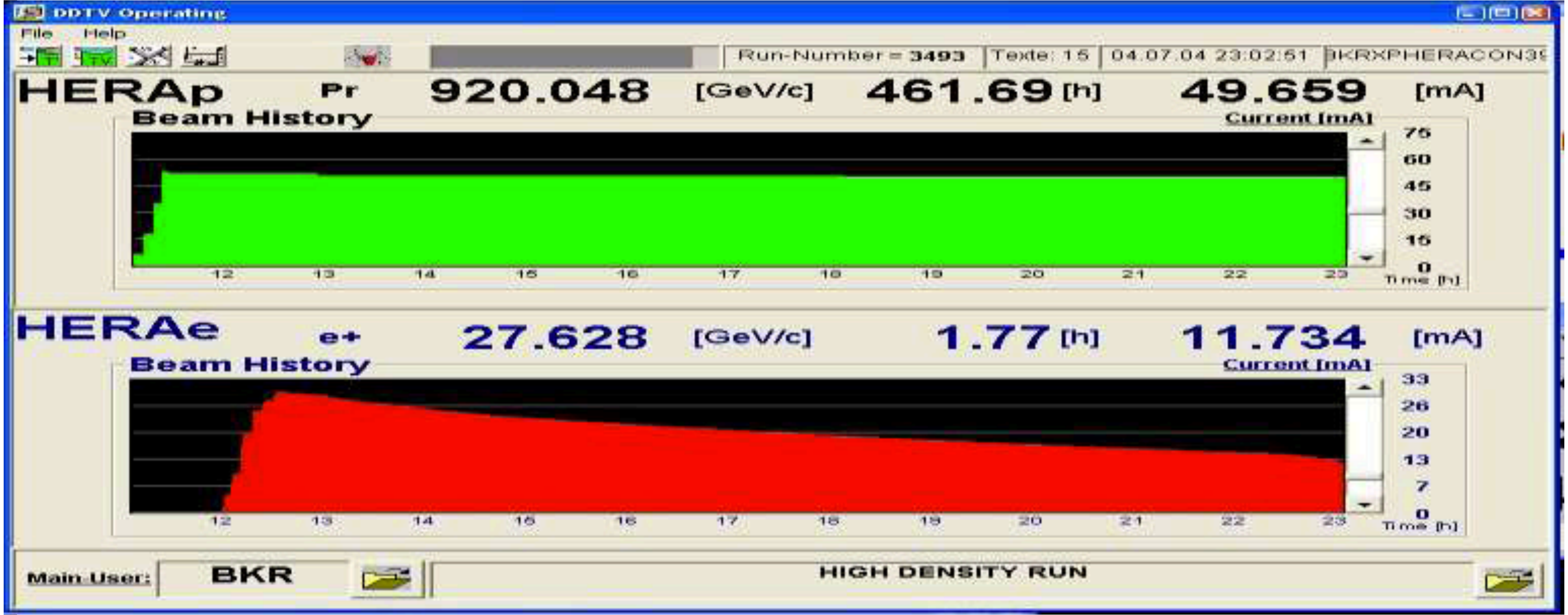
HERA



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What is beam dynamics?

We do beam dynamics to understand the motion of particles in linear and circular accelerators, to

- Understand the fundamentals of existing machines
- Optimise and commission accelerators
- Design new machines e.g. a new collider
- Design novel machines e.g. a non-scaling FFAG

How do we do this?

The fundamental tool of a person ' doing' beam dynamics is knowing how to calculate the motion of a charged particle in a real electromagnetic field

- Magneto-static configurations
- Time dependent fields
- 'Optics'
- What are the approximations used?
- How do the particles interact with the surroundings?
- Do the particles in a bunch interact with themselves?

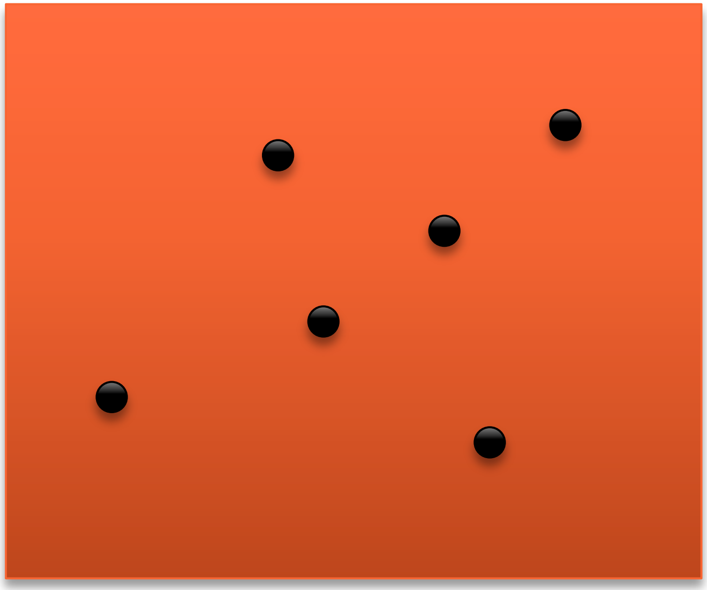
The most basic question is how do I represent the beam passing through the accelerator in my beam dynamics language? This leads us to a hierarchy of beam descriptions

A hierarchy of beam descriptions I

Think of a box of gas…

We can think of this system as being described by several numbers

The pressure (P)
The temperature (T)
The volume (V)
The number of moles (n)



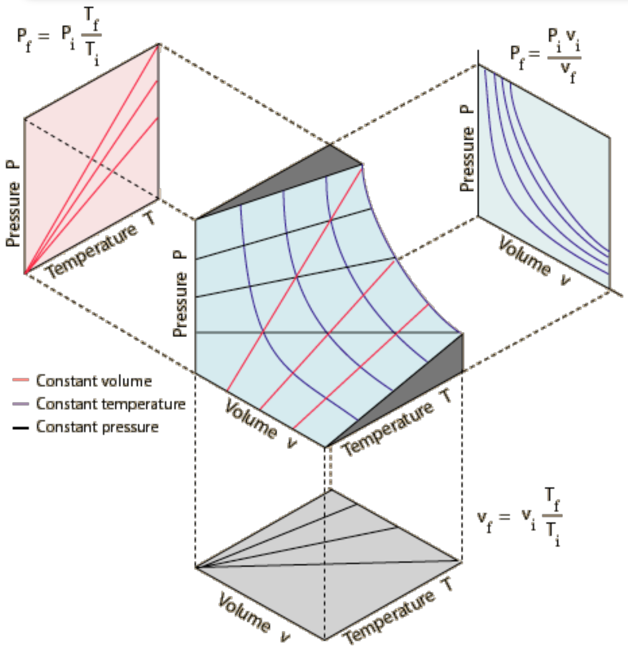
An equation of state relates these quantities together

An ideal gas obeys the ideal gas law

$$PV=nRT$$

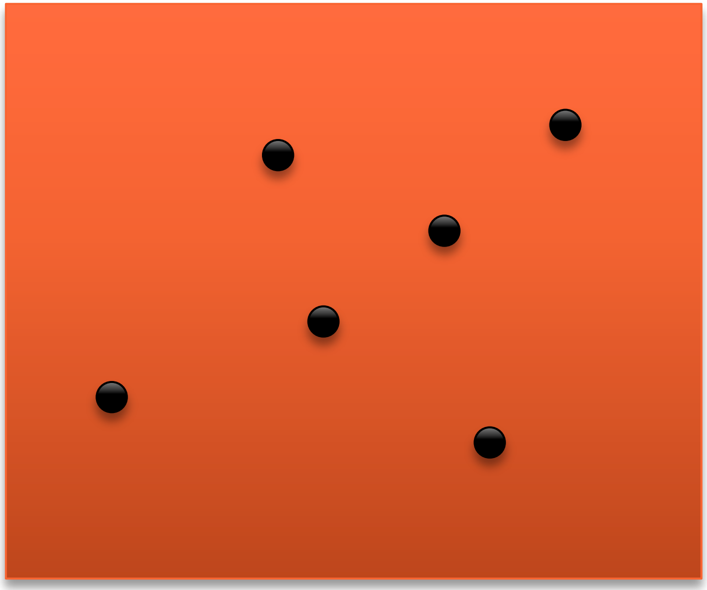
Which relates the state variables to each other and tell us how they change.

e.g. an isobaric change of state



A hierarchy of beam descriptions II

Our gas is also made up of a collection of gas molecules, each with a position and a momentum in every degree of freedom of the system



Each molecule has a speed and a kinetic energy (translational energy)

This is a microscopic view of our gas in a box, and an equally valid view

The two pictures are related.....

$$K_{\text{tr}} = \frac{3}{2} nRT$$

So K_{tr} (the average kinetic energy) is directly proportional to the macroscopic temperature of the gas

Global vs. local dynamics

So it's common in physics to have several different, but equivalent, views of the same physical situation

- Physics of an ideal gas
- Quantum mechanics, with wave and matrix formulations

We have the same situation in beam dynamics

<u>Global view</u> where we assume a ring or beam line exists and study the global properties of the system. For example, stability or tune

<u>Local view</u> where we worry about the details of the machine

- What frame of reference is best?
- What do the fields of our magnets look like in this frame?
- How do I stitch together the the frames of many magnets?
- How does a single particle move in this system?

Equivalent ways of doing dynamics

So we have different ways of looking at a beam

There are also several ways of ‘doing’ particle dynamics

These ways are equivalent to each other and all can be used to solve dynamical problems. The three formulations of dynamics are

1) Newtonian dynamics
2) Lagrangian dynamics
3) Hamiltonian dynamics

The one you should choose depends on the kind of problem you are solving

In accelerator physics we tend to use Newtonian and Hamiltonian dynamics, and each one has its own merits.

Let’s look at each of them in turn…

Newton

Newton's formulation involves relating a force to the resulting motion using Newton's 2nd law

$$\frac{d}{dt}m\dot{\vec{x}} = F(\vec{x}, \dot{\vec{x}}, t)$$

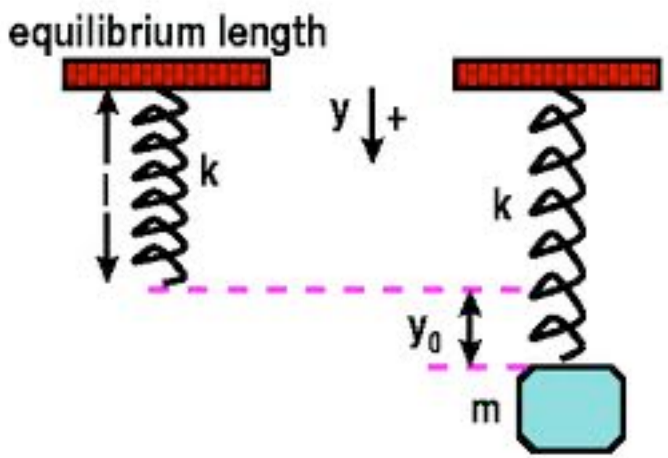
Doing the physics means working very hard to figure out the force F, which depends on the particle positions, velocities and on the independent variable time (t)

We then solve Newton's second law to figure out the evolution of the position and velocity of a particle as a function of the independent variable t

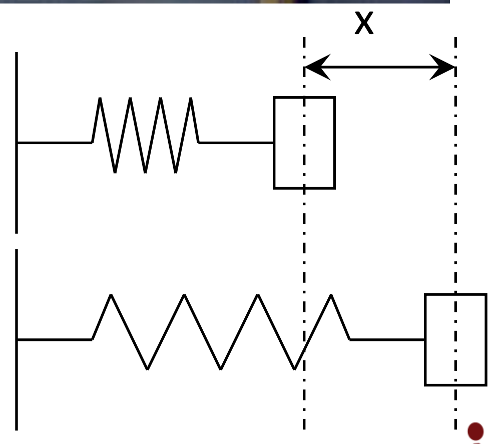
A simple harmonic oscillator

Think of a mass on a spring…

Mass m, spring constant k



Here, the force on the mass from the spring pushes the mass back to the equilibrium point



$$F = -kx$$

A simple harmonic oscillator

The force is given by

$$F = -kx$$

When we use Newton's law we get the equation of motion

$$m\frac{d}{dt}\dot{x} = -kx \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This standard equation has oscillatory solution (with two constant)

$$x(t) = x_0 \sin(\omega t + \phi_0) \qquad \omega^2 = \frac{k}{m}$$

Which we can write as

$$x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Note we can write the equation of motion as

$$m\frac{d}{dt}\dot{x} = -m\omega^2 x$$

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A simple harmonic oscillator with the opposite force

Here, the force is given by

$$F = +kx$$

Which leads to the standard equation of motion

$$\frac{d^2x}{dt^2} = +\frac{k}{m}x$$

Note the force is now pushing the particle to larger amplitude, and we can write the solution as (again with two constants)

$$\omega^2 = \frac{k}{m} \qquad x(t) = c_1 \cosh(\omega t) + c_2 \sinh(\omega t)$$

The Lorentz force law

A very important equation for accelerator physics is the Lorentz force law.

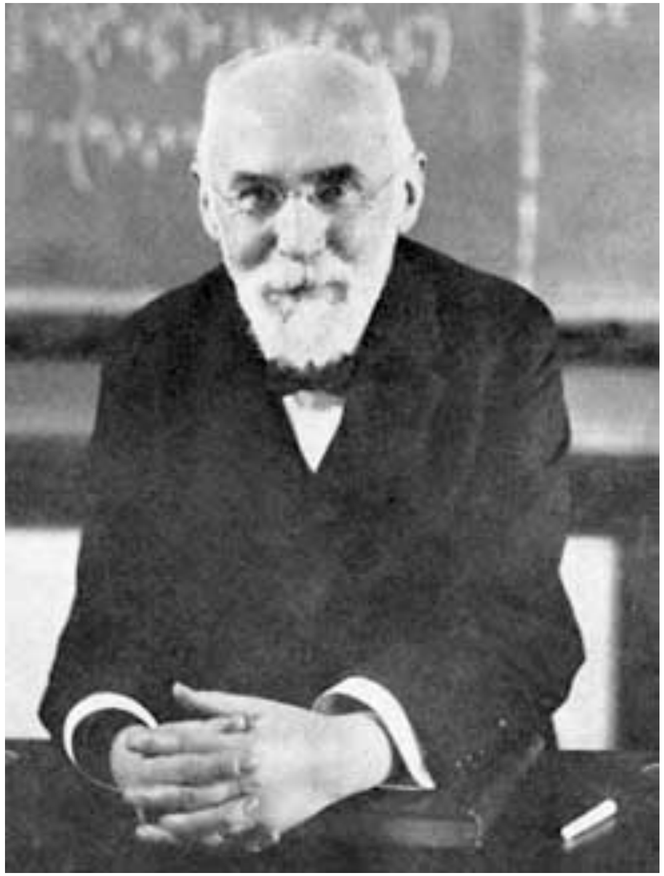
This relates the force on a particle of charge q from an electric and magnetic field

$$\vec{F} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$$

The vector nature of this equation is very important, and was covered in courses on electromagnetism.

E and **B** themselves obey equations, which specify their vector fields in the presence of sources and boundary conditions.

Usually the speed is close to the speed of light and so magnetic fields are efficient to guide particles



Generally v ~ c in a high energy storage ring. But not always!

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Hamiltonian mechanics

Hamiltonian mechanics is a formulation of dynamics by Hamilton.

This way of working carries the advantage of a well defined mathematical framework, where keeping control of variables, invariants and approximations is much easier than when using Newtonian dynamics.

Of central importance is the Hamiltonian function H, which is a function of a set of conjugate variable pairs.

For example, in one dimension we would deal with the position and the canonical momenta of that particle as a conjugate pair

Hamilton’s equations

Given a Hamiltonian which is a function of the canonical variables, the dynamics are worked out using Hamilton’s equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

This gives a set of 2n first order ODEs to solve, where n is the number of dimensions of the system

Conceptually the Hamiltonian plays the role that the force does in Newtonian dynamics.

Conventionally, students of accelerator physics learn the way with Newton first and the way of Hamilton second.

In this course we’ll largely adopt this conventional approach, of using forces first.

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A simple example – a harmonic oscillator

The Hamiltonian for an harmonic oscillator is given by

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

In terms of the particle position and canonical momenta. Hamilton's equations directly give

$$\frac{dx}{dt} = \frac{p_x}{m}$$
$$\frac{dp_x}{dt} = -m\omega^2 x$$

Which are equivalent to the equations of motion obtained from Newton's formulation of dynamics. Note the Hamiltonian can be written as the sum of the kinetic and potential energies

$$H = T + V$$

The Hamiltonian is a conserved quantity

If we take the total time derivative of the Hamiltonian which depends on x and p_x

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p_x}\frac{dp_x}{dt} + \frac{\partial H}{\partial t}$$

We can use Hamilton's equations to rewrite some of the derivatives

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{dp_x} - \frac{\partial H}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t}$$

If the Hamiltonian does not depend explicitly on time, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

And so H is a conserved quantity of the system. This is very powerful

Hill's equation

So we've seen that simple harmonic oscillator equation, with different signs for the spring constant, can lead to oscillating and diverging solutions.

The basic equations of motion for a particle in an accelerator are called Hill's equations, which we will soon derive to be

$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$

$$z'' - kz = 0$$

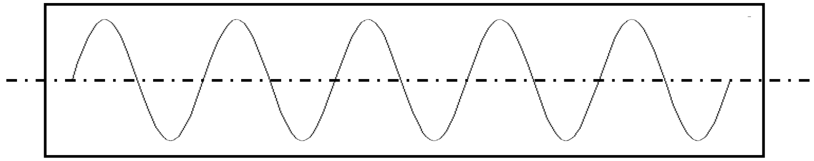
Take the second one, which is the equation of motion for a region with no bending. Depending on the sign of the constant this corresponds to either sine/cosine or sinh/cosh solutions for z(s).

So it seems that we can think of an accelerator, at least linearly, as being made up of piece-wise regions with different sizes and signs of the spring constant.

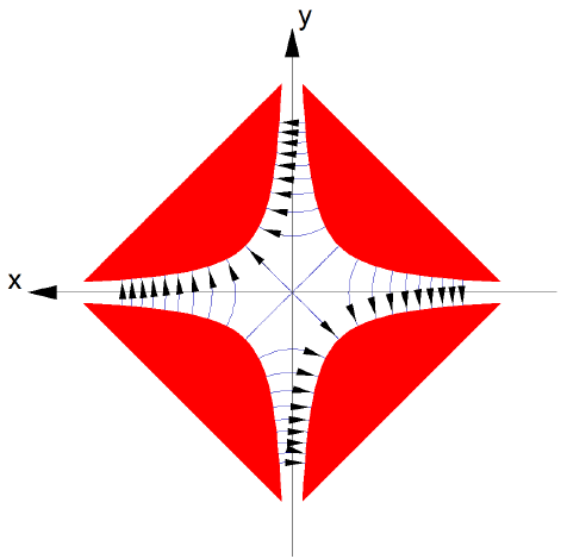
$$x(s)'' + \left(k(s) + \frac{1}{\rho(s)^2}\right) x(s) = 0$$

A mass on a spring with variable k == optical system

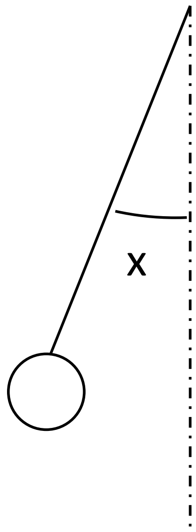
SHM through a long quadrupole



$$\frac{d^2u}{ds^2} + Ku(x) = 0$$



Normal quadrupole
$B_x = b_2 \frac{y}{r_0}, \quad B_y = b_2 \frac{x}{r_0}.$



Physical analogies

Pendulum
(small angles only!)

The basic idea

Basically we shall use magnetic and electric fields to accelerate and control the particles inside our accelerator

$$\vec{F} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$$

We use magnetic fields to deflect particles as we really benefit from
The presence of the velocity in the Lorentz force law. Consider a field

$$B = 1\,\mathrm{T}$$

The magnetic force on a particle of velocity c and charge q can be written as

$$F_B = q \times 3 \times 10^8 [m/s] \times 1[Vs/m^2]$$

Which we can write in terms of an equivalent electric field (which is high!)

$$F_B = q \times 300[MV/m]$$

Given that making electric fields about 1 MV/m is hard, we really benefit by using magnetic fields to deflect our particles.
We will, however, use electric fields to accelerate our particles (why?)

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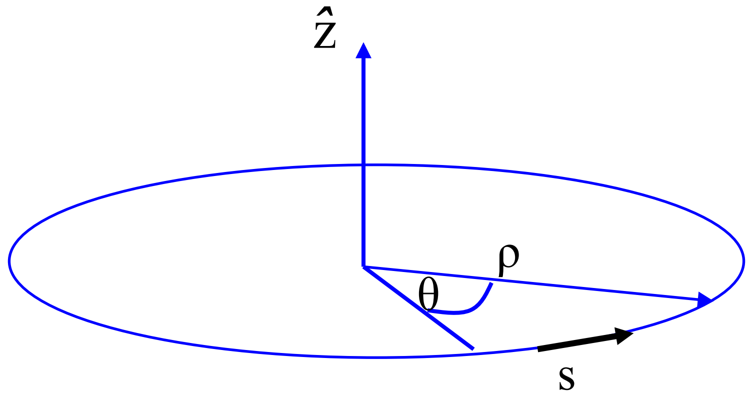
A circular orbit and the design orbit

We want to make our particles move in a circle, so we apply a constant magnetic field.



circular coordinate system

The force on the particle is always at right-angles to the motion and given by

$$F_B = qvB$$

We equate this to the centripetal force

$$F = \frac{\gamma m v^2}{\rho}$$

$$\Rightarrow \quad qB = \frac{\gamma m v}{\rho} \quad \Rightarrow \quad B\rho = \frac{p}{q}$$

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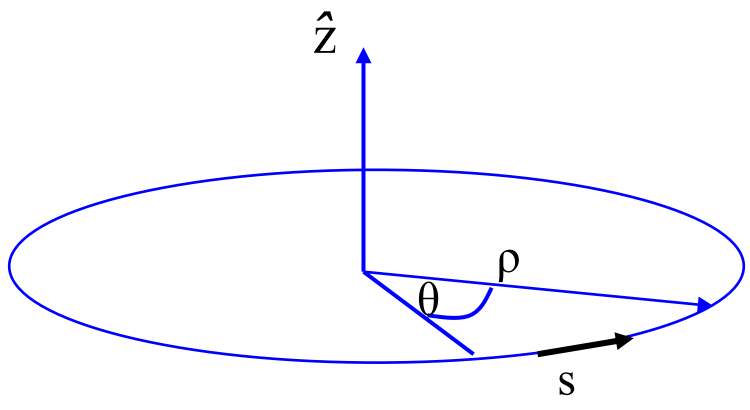
A circular orbit and the design orbit

$$B\rho = \frac{p}{q}$$



circular coordinate system

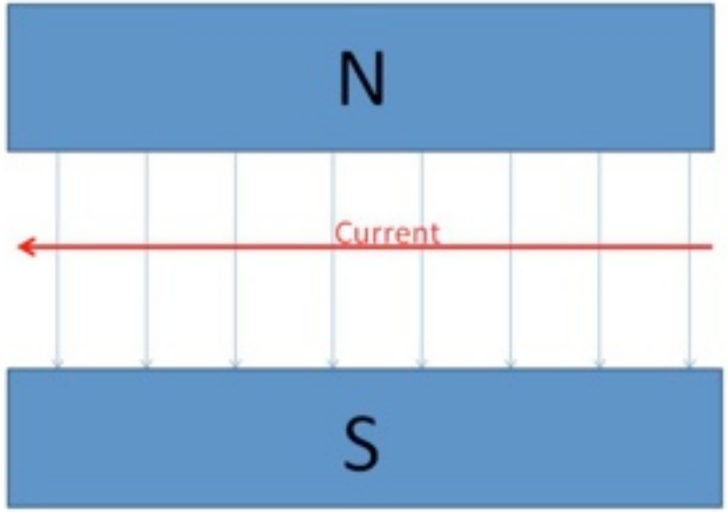
So a particle with momentum p and charge q in a guide field B will follow a circular path of radius ρ.

Or, the product of the field and the radius is a function only of the particle momentum and charge. The particular combination p/q is exactly that which appears in the normalisation of many physical quantities in beam dynamics.

The quantity Bρ, called the beam rigidity, is often used instead of p/q. Note that the beam rigidity does not refer to a specific field strength or radius of curvature of the trajectory. It should be thought of as simply another way of writing the reference momentum.

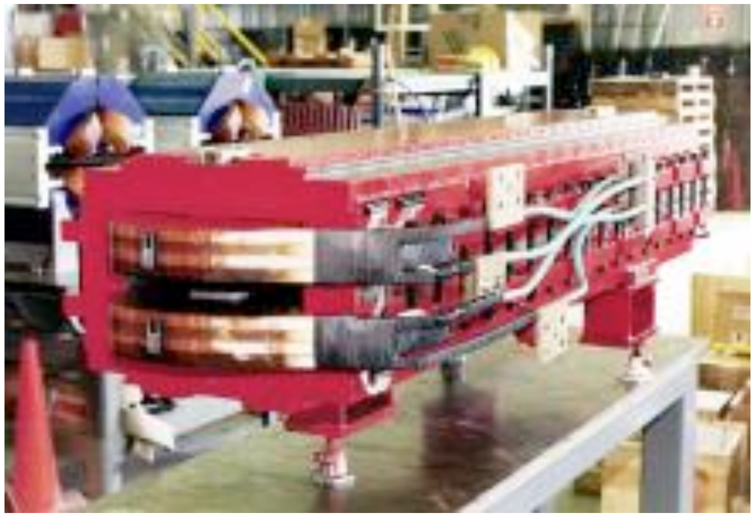
The magnetic guide field

$$B\rho = \frac{p}{q} \rightarrow \frac{1}{\rho} = \frac{qB}{p}$$



$$B = 8.3T$$

$$p = 7000\ GeV/c$$



$$\frac{1}{\rho} = q\frac{8.3[\mathrm{Vs/m^2}]}{7000 \times 10^9[\mathrm{eV/c}]} = \frac{8.3\,\mathrm{s} \times 3 \times 10^8[\mathrm{m/s}]}{7000 \times 10^9[\mathrm{m^2}]}$$

$$\frac{1}{\rho} = 0.33\frac{8.3}{7000}[/\mathrm{m}] = 3.9 \times 10^{-4}[/\mathrm{m}]$$

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Beam rigidity

For the case (normally encountered) of

$$E_0 >> m\,c^2$$

We can calculate the beam rigidity from the easy-to-remember equation

$$B\rho[\mathrm{Tm}] = \frac{E_0[\mathrm{eV}]}{c[\mathrm{m/s}]}$$

This can also be written as

$$B\rho[\mathrm{Tm}] = 3.3\ p_0[\mathrm{GeV/c}]$$

$$\frac{1}{\rho} = 0.2998 \times \frac{B[\mathrm{T}]}{p_0[\mathrm{GeV/c}]}$$

Which is useful when working with high energy colliders, when the beam energy is normally expressed in units of GeV

We will generally normalise magnetic field strengths to the beam momenta, and obtain energy-independent ‘k values’

$$k_0 = \frac{q}{p_0} B$$

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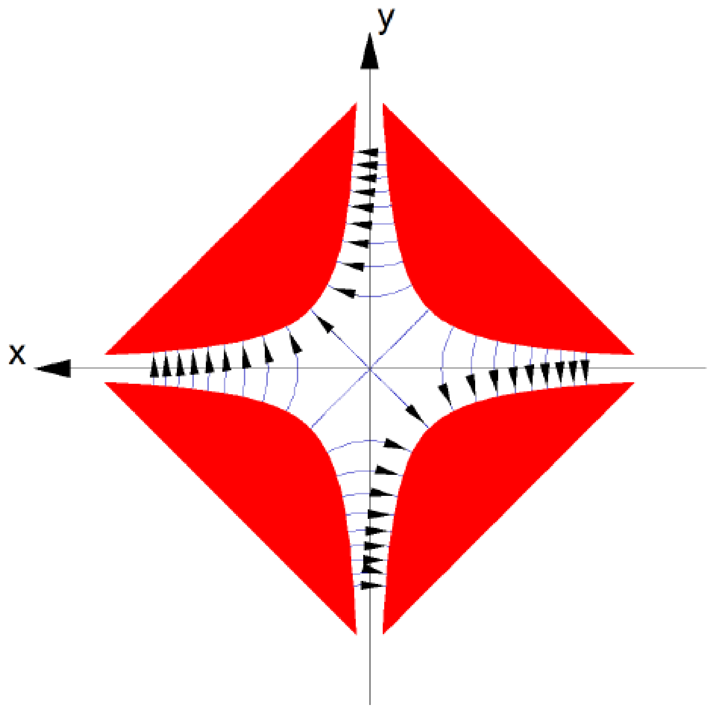
A quadrupole



A linearly increasing field (from the centre of the magnet)

Normal quadrupole

$$B_x = b_2 \frac{y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}.$$

$$k_1 = \frac{q}{P_0} \frac{b_2}{r_0}$$

Normalised field gradient
(this is the number used in nearly all codes)
Doesn't depend on energy

$$\vec{B} = g(z\vec{x} + x\vec{z})$$

$$g = \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

Notice $\nabla \times \vec{B} = 0$

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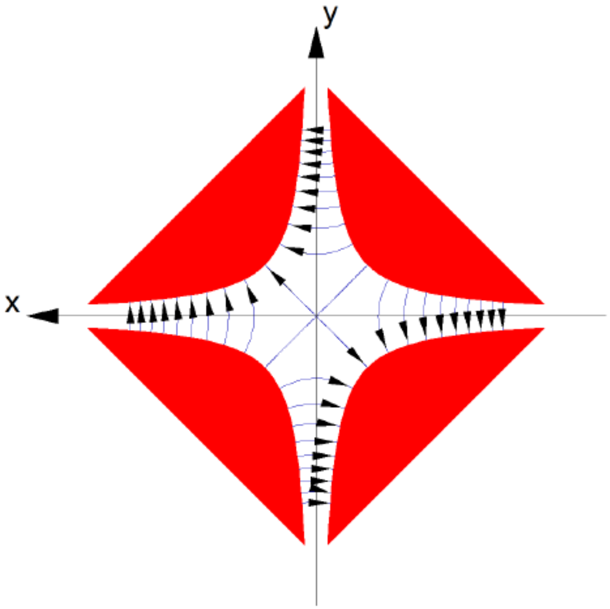
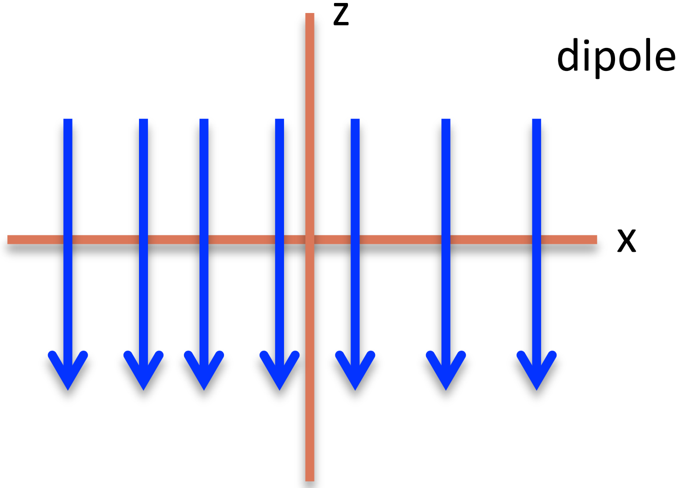
A quadrupole

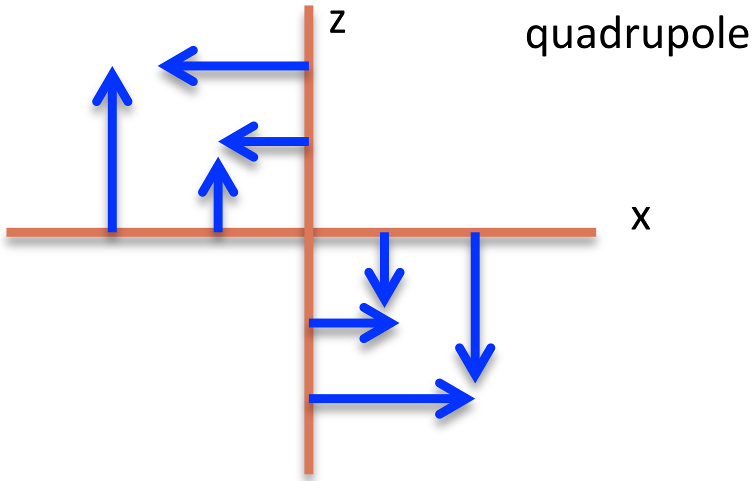


Normal quadrupole

$$B_x = b_2 \frac{y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}.$$

$$\vec{B} = g(z\vec{x} + x\vec{z})$$





For motion into the page, this quadrupole gives a focusing force in x. However it defocuses in z, due to Maxwell's equations

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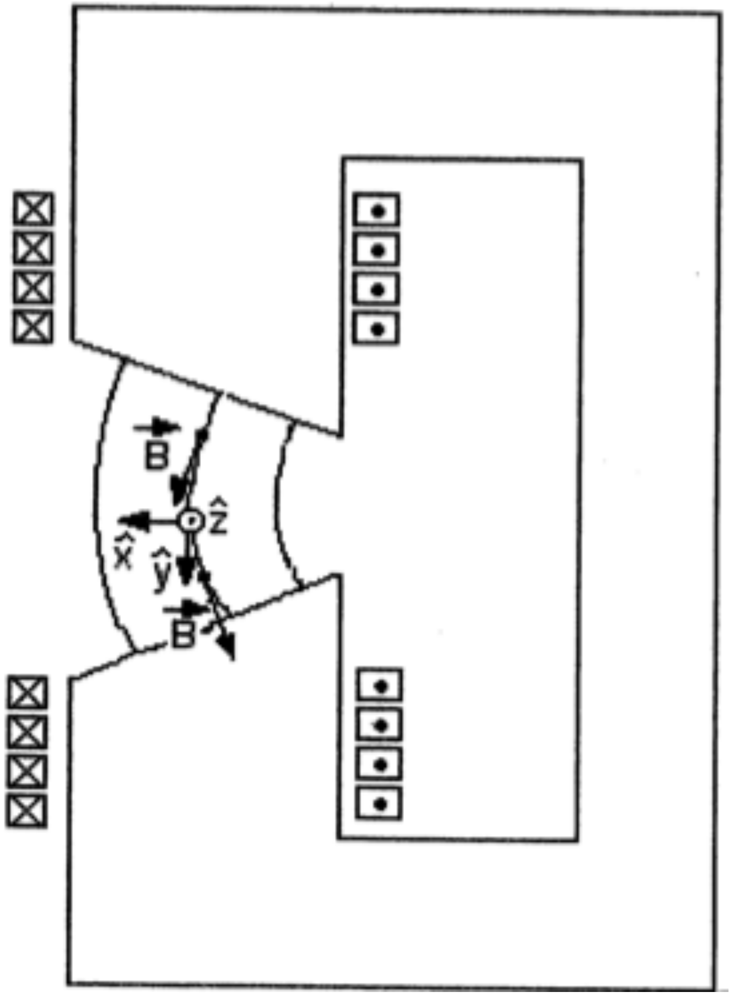
A combined function magnet



We can also provide the magnetic field gradient needed for beam focusing by tilting the magnetic pole faces.

Notice that there is a component of B such that B_x is proportional to y

So this magnet has both a dipole and a quadrupole component. A **combined function magnet.**

(The pole face tilt must be small for stable motion…weak often called weak focusing)

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Exercise – parameters of storage ring

Imagine a proton storage ring, with a beam momentum of 20 TeV/c and 17000 proton bunches, with 1.E10 protons per bunch.

1) What is the stored energy of this machine per beam?

2) If the circumference is 83km, and the field is 6.6 T, what fraction of ring is filled with dipoles?

3) We would like to build a high energy LHC, with a beam momentum 16 TeV/c. What is the big challenge here?

4) The LHC beam energy is 360 MJ. What problems might this cause?

The expansion of the fields

We will now figure out the equations of motion for a particle in an accelerator

1) We will figure out the motion of the particles w.r.t. a design orbit

2) We will assume the deviation of particle coordinates from this design orbit is small, so x,z << the bending radius

(2) means that we only need to take into account linear terms in the deviation of the field B w.r.t. x and z

So it makes sense to take some arbitrary guide field and make a Taylor expansion around the design orbit and normalise to the beam rigidity

$$B_z(x) = B_{z0} + \frac{dB_z}{dx}x + \frac{1}{2}\frac{d^2B_z}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_z}{dx^3}x^3 + \dots$$

$$\frac{e}{p}B_z(x) = \frac{e}{p}B_{z0} + \frac{e}{p}\frac{dB_z}{dx}x + \frac{e}{p}\frac{1}{2}\frac{d^2B_z}{dx^2}x^2 + \frac{e}{p}\frac{1}{3!}\frac{d^3B_z}{dx^3}x^3 + \dots$$

$$\frac{e}{p}B_z(x) = \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}ox^3 + \dots$$

Oscillations and focusing

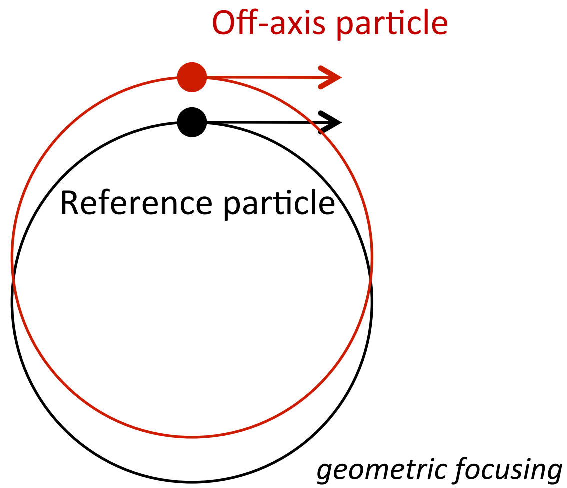
Consider a constant energy particle in a fixed dipole field (into the page)

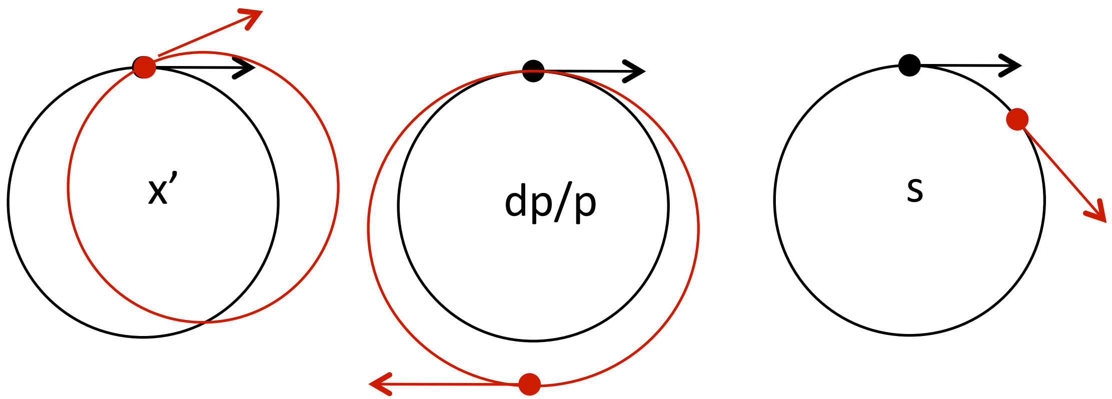
Reference particle executes cyclotron motion

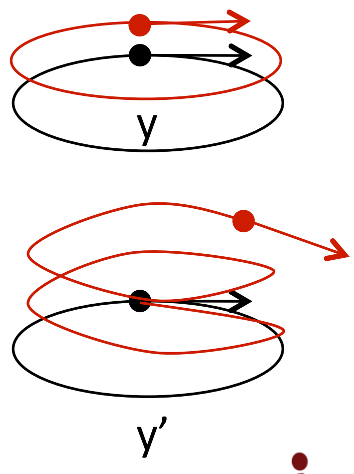
Other particles of the same energy execute cyclotron motion of same radius

Compared to reference trajectory, this looks like an **oscillation.** We'll call this **a betatron oscillation.**

Number of oscillations per turn (the 'tune') is 1.







PART TWO- HILL’S EQUATION

The equations of motion in an accelerator

We will now derive the transverse equations of motion for a particle moving under the influence of dipole and quadrupole fields in an accelerator. These are called Hill's equations. The steps we will follow are

1) Define our curved coordinate system and our coordinates

2) Work out what position, velocity and acceleration vectors look like in this coordinate system. Note we are concerned with the transverse motion and not concerned with longitudinal motion at this stage.

3) Use the Lorentz force law to figure out the forces on the particle and write down an expression for the change in particle momentum

4) Change the independent variable from time to space

5) Expand out the equations so they are **linear** in x and z (this means we can solve them)

6) Specialise to the case of pure dipole and quadrupole fields

The result will be a second order ordinary differential equation in each plane, for the motion of the particle in the machine. Hill's equations.

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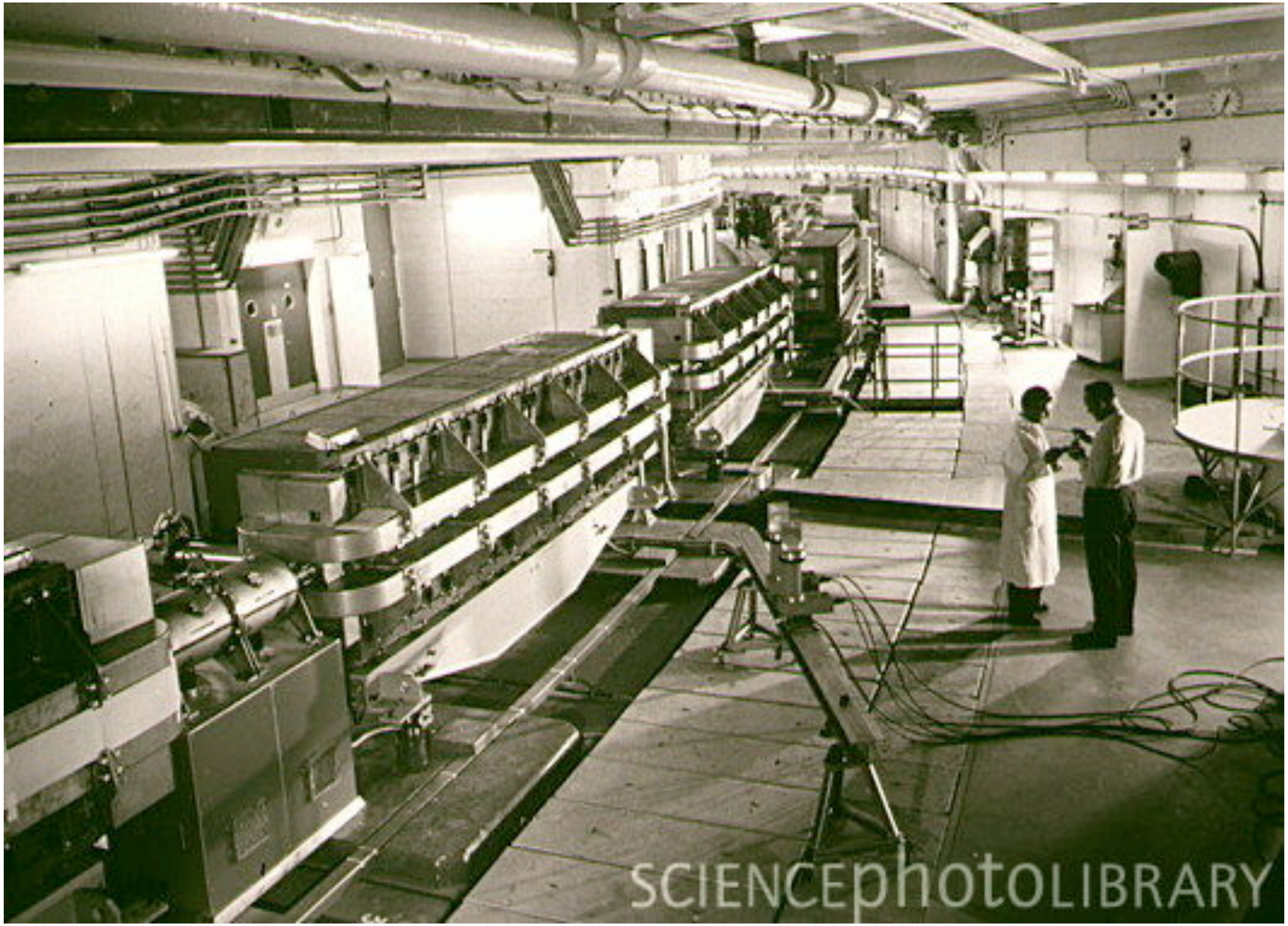
A storage ring, with a design orbit and dipoles to define it



Our coordinate system and our coordinates (x, z, s)

We will now develop the equations of motion in a linear or circular machine.

We'll use a curved coordinate system, with this curvature produced by a local dipole field.

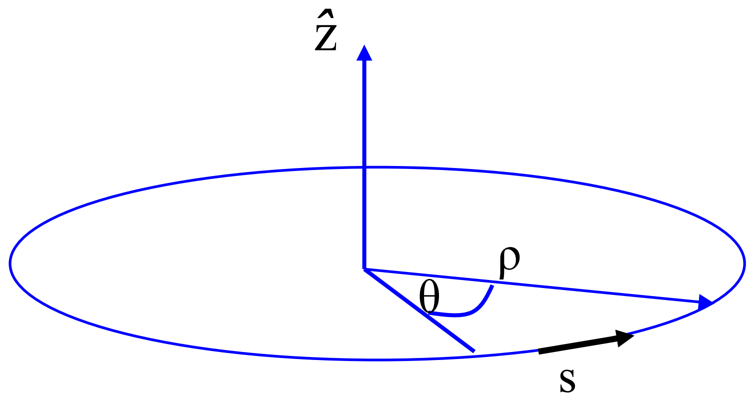
The local curvature is ρ
The path length along this design trajectory will be s.

The position vector of a particle in this coordinate system is

$$\vec{R} = r\vec{x} + z\vec{z}$$

where

$$r = \rho + x$$



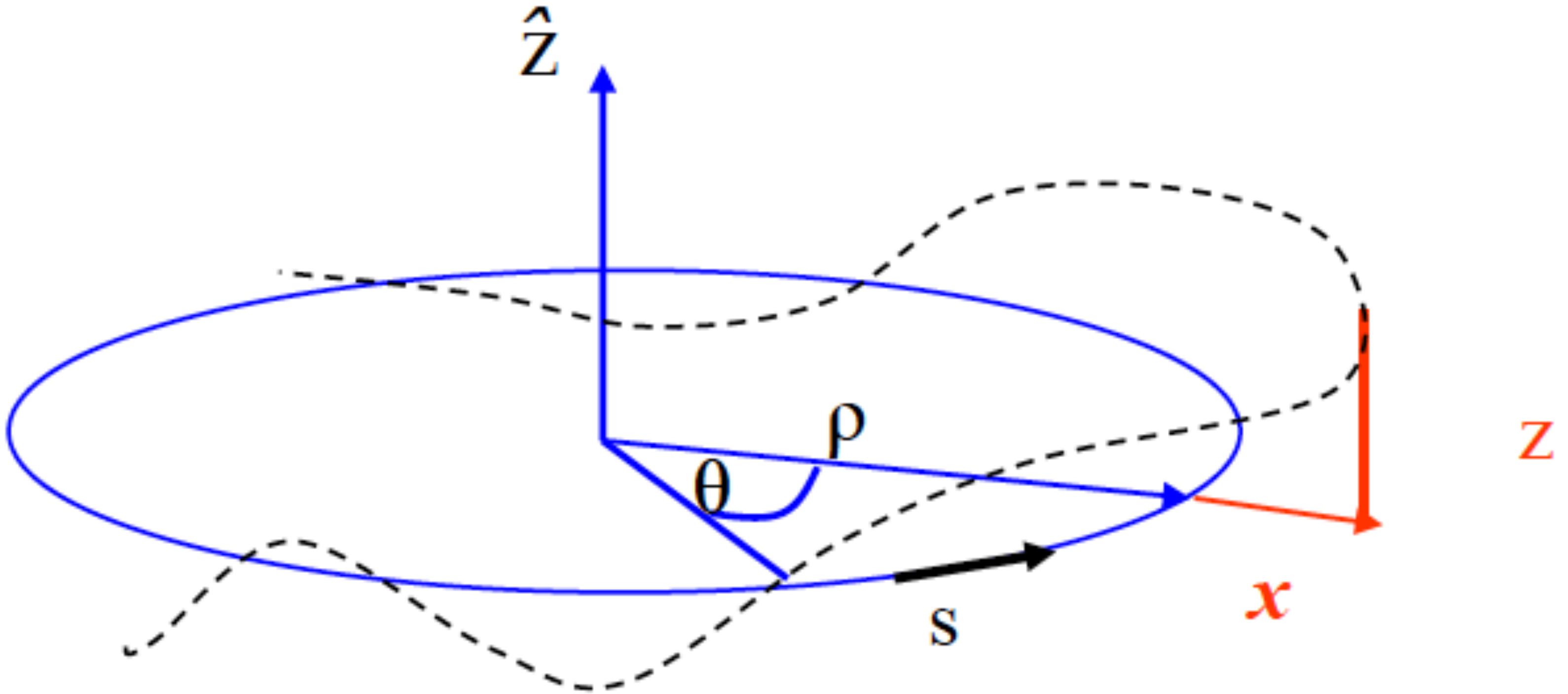
circular coordinate system

The curved reference trajectory is normally called the orbit, and the coordinate system moves with a reference particle around the design orbit defined by the **dipoles**.

So our coordinates represent deviations with respect to the design (**ideal**) orbit, and we assume these deviations will be small (x is normally around millimetres).

For coordinates relative to this design orbit we use the position and slope dx/ds=x'

Coordinates are with respect to the design orbit



The design orbit has curvature ρ

The path length along this orbit will ultimately be labeled by s

Although we start off with time (t) as the independent variable

We will generally ignore the longitudinal component of motion

At any point we have a right-handed coordinate system (x, z, s)

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Kinematics in this coordinate system

Let's do some kinematics in this coordinate system. The velocity is given by

$$\vec{R} = r\vec{x} + z\vec{z} \qquad \vec{v} = \dot{\vec{R}}$$

Changes around ring

$$\vec{v} = \dot{r}\vec{x} + r\dot{\vec{x}} + \dot{z}\vec{z}$$

tangential velocity

$$\vec{v} = \dot{r}\vec{x} + r\dot{\theta}\vec{s} + \dot{z}\vec{z}$$

So $v_x = \dot{r} \quad v_y = \dot{z} \quad v_s = r\dot{\theta}$

$$r = \rho + x$$

We can differentiate again for the acceleration vector

$$\ddot{\vec{R}} = \ddot{r}\vec{x} + \dot{r}\dot{\vec{x}} + \dot{r}\dot{\theta}\vec{s} + r\ddot{\theta}\vec{s} + r\dot{\theta}\dot{\vec{s}} + \ddot{z}\vec{z}$$

$$\ddot{\vec{R}} = \ddot{r}\vec{x} + 2\dot{r}\dot{\theta}\vec{s} + r\ddot{\theta}\vec{s} - r\dot{\theta}^2\dot{\vec{x}} + \ddot{z}\vec{z}$$

Note we used $\dot{\vec{x}} = \dot{\theta}\vec{s} \quad \dot{\vec{s}} = -\dot{\theta}\vec{x}$

The Lorentz force law

To figure out the force on our particle, we use the Lorentz force law with no electric field and only transverse magnetic fields,

$$\frac{d\vec{p}}{dt} = e\vec{v} \times \vec{B}$$

Expanding the cross product gives the components of the rate of change of p

$$\frac{d\vec{p}}{dt} = e \begin{vmatrix} \vec{x} & \vec{z} & \vec{s} \\ v_x & v_z & v_s \\ B_x & B_z & 0 \end{vmatrix} = -ev_sB_z\vec{x} + ev_sB_x\vec{z} + e(v_xB_z - v_zB_x)\vec{s}$$

Writing the momentum in terms of our position vector **R**

Can you prove this?

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}m\gamma\dot{\vec{R}} = m\gamma\ddot{\vec{R}}$$

(assuming gamma is constant, as we only have magnetic fields)

We get an expression for the rate of change of p component by component

$$\frac{d\vec{p}}{dt} = m\gamma(\ddot{r}\vec{x} + 2\dot{r}\dot{\theta}\vec{s} + r\ddot{\theta}\vec{s} - r\dot{\theta}^2\vec{x} + \ddot{z}\vec{z})$$

Equating, the radial and vertical equations of motion then read

We've dropped the s parts

$$\ddot{r} - r\dot{\theta}^2 = -\frac{eB_y}{m\gamma}v_s \qquad \ddot{z} = \frac{eB_x}{m\gamma}v_s$$

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Changing the independent variable from t to s

We now want to change the independent variable from time to s, as this is what we care about and magnets are localised in position and not time. It's wrong to use

$$\frac{ds}{dt} = v_s$$

We actually need to include an extra factor due to geometry of the orbit

(See Lee)

$$\frac{ds}{dt} = v_s \frac{\rho}{\rho + x}$$

(need to scale back from orbit to reference orbit, as it's the latter we want for ds/dt)

We can now convert the the radial and vertical derivatives to get

$$\dot{r} = \frac{dr}{dt} = \frac{v_s \rho}{\rho + x}\frac{dr}{ds} = \frac{v_s \rho}{\rho + x}\frac{dx}{ds}$$

and so

$$\ddot{r} = \frac{d^2 r}{dt^2} = \left(\frac{v_s \rho}{\rho + x}\right)^2 \frac{d^2 x}{ds^2}$$

and

$$\ddot{z} = \left(\frac{v_s \rho}{\rho + x}\right)^2 \frac{d^2 z}{ds^2}$$

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The equations of motion

We can use the equations to change the independent variable from t to s in the equations of motion we derived from the Lorentz force law.
We do this by swapping each dt for a ds, and then use the fact that

$$r\dot{\theta} = v_s \qquad \Delta S = \rho\Delta\theta \qquad r\dot{\theta}^2 = \frac{r^2\dot{\theta}^2}{r} = \frac{v_s^2}{r} = \frac{v_s^2}{\rho + x}$$

Doing this we then obtain the equation for x (and similarly for z)

$$\left(\frac{\rho}{\rho + x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho + x} = -\frac{eB_y}{m\gamma v_s}$$

$$\left(\frac{\rho}{x+\rho}\right)^2 \sim 1 - \frac{2x}{\rho} + \ldots$$

$$\frac{1}{\rho + x} \sim \frac{1}{\rho} - \frac{x}{\rho^2} + \ldots$$

We then expand the functions of x and z, keeping only first order. After some algebra we obtain the equations of motion in x and z

$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} = -\frac{e(B_z - B_{z0})}{p} \qquad \frac{d^2z}{ds^2} = \frac{eB_x}{p} \qquad \frac{1}{\rho} = B_{z0}\frac{e}{p}$$

Note we assumed $\quad p = m\gamma v \sim m\gamma v_s$

And we have normalised the field to p/e

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Equations of motion in dipole and quadrupole fields

An important application of our transverse equations of motion is when the magnetic field contains only dipole and quadrupole components, so we write

$$\vec{B} = B_{z0}\vec{z} + g(z\vec{x} + x\vec{z}) \qquad g = \partial B_z/\partial x = \partial B_x/\partial z$$

If we do this, the guide field term B_{z0} cancels, and we are left with the equations of motion in dipole and quadrupole fields in both planes

$$\frac{d^2x}{ds^2} + \left(\frac{g}{B\rho} + \frac{1}{\rho^2}\right)x = 0 \qquad \frac{d^2z}{ds^2} - \frac{g}{B\rho}z = 0$$

Note that the fields B_{z0} and g are periodic functions of the independent variable s. This periodicity is either the machine circumference or the length of a repeating period L.

So we have derived our equations of motion for transverse motion.
Note we focus in x and defocus in y : as curl B=0

(Note some authors play with the signs of the pieces in the second term of the equation for x. For example, Wille writes –k. Be careful!)

Exercises

1. Prove the magnetic field bends the trajectory of the particle but does not do any work on the particle. This means the particle energy does not change.

Hint : d(gamma)/dt=0

2. Why are magnetic forces necessarily transverse? What does this mean about longitudinal control of a beam?

So we have Hill's equations

So, after some effort, we derived Hill's equations. They are linearised second order differential equations for the transverse variables x and z in dipole and quadrupoles fields in a particle accelerator.

$$\frac{d^2x}{ds^2} + \left(\frac{g}{B\rho} + \frac{1}{\rho^2}\right) x = 0 \qquad \frac{d^2z}{ds^2} - \frac{g}{B\rho} z = 0$$

In the equation for x, g comes from the quadrupoles and there is also natural focusing from the dipoles in the plane of curvature.
We assumed that k and rho are constant in the solution, so we assume these functions of s are piecewise **constant**

$$x(s)'' + \left(k(s) + \frac{1}{\rho(s)^2}\right) x(s) = 0 \qquad \begin{aligned}\rho &= \text{const}\\ k &= \text{const}\end{aligned}$$

We can compactly write Hill's equations in both planes, denoting x or z with u and writing the position dependent 'spring constant' as K

$$\frac{d^2u}{ds^2} + Ku(x) = 0 \qquad \begin{aligned}K &< 0 \text{ for } u = z\\ K &> 0 \text{ for } u = x\end{aligned} \qquad k = \frac{g}{B\rho}$$

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So we have Hill's equations

So we have the transverse equation of motion

$$\frac{d^2u}{ds^2} + Ku(x) = 0 \qquad\qquad k = \frac{g}{B\rho}$$

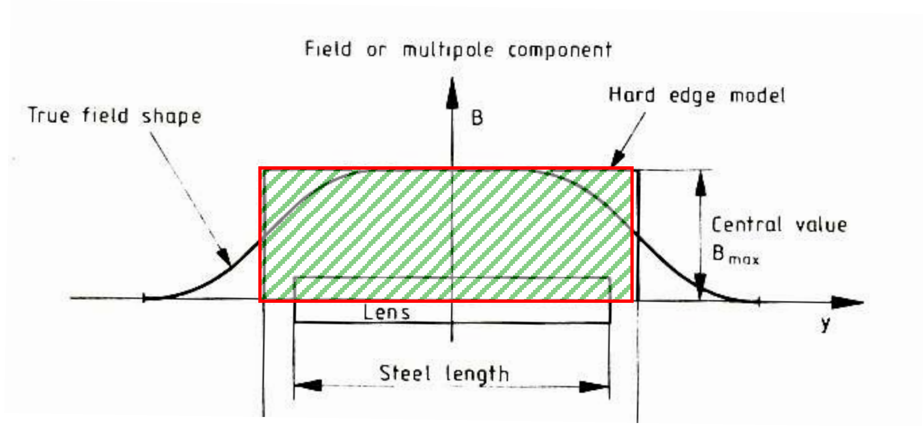
With the periodic functions of s describing the lattice

$$K_x = \frac{1}{B\rho}\frac{\partial B_z}{\partial x} + \frac{1}{\rho^2}$$

$$K_y = -\frac{1}{B\rho}\frac{\partial B_z}{\partial x}$$

In these equations, ρ comes from the natural horizontal focusing in dipoles and the gradient is the strong focusing in quadrupoles.

Effective magnet lengths



In reality, magnets tend to have fields which extend beyond the physical length of the magnet. We account for this using an effective length, which is the length of the magnetic elements *as seen by the beam.*

$$B \times l_{eff} = \int_0^{l_{\mathrm{mag}}} B ds$$

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Solution of piece-wise Hill's equations

$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$

$$z'' - kz = 0$$

These are our equations of motion, the horizontal plane and the vertical plane

$$x'' + K \cdot x = 0$$

$$K = k \quad K = k + \frac{1}{\rho^2}$$

These equations look like the equation for a simple harmonic oscillator, if we write K for the constant and assume it's constant over the region of the accelerator we are interested in

So we know the solution from out studies of a mass on a spring! Let's guess at

$$K > 0 \qquad x(s) = c_1 \cos(\sqrt{K}s) + c_2 \sin(\sqrt{K}s)$$

Take the derivatives and substitute into the equation of motion

$$x'(s) = -c_1\omega \sin(\sqrt{K}s) + c_2\omega \cos(\sqrt{K}s) \qquad \sqrt{K} = \omega$$

$$x''(s) = -c_1\omega^2 \cos(\sqrt{K}s) - c_2\omega^2 \sin(\sqrt{K}s) = -\omega^2 x(s)$$

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Solution of piece-wise Hill's equations

So this solution seems to work. Of course we assumed K > 0 to get an oscillatory solution but let's worry about the case of K < 0 later.

What about those two unknown constants? We can fix these from the initial conditions

$$x(0) = x_0 \rightarrow c_1 = x_0$$

$$x'(0) = x'_0 \rightarrow c_2 = \frac{x'_0}{\sqrt{K}}$$

So we can write down the evolution of the variables x and x' in a region with constant and positive K as

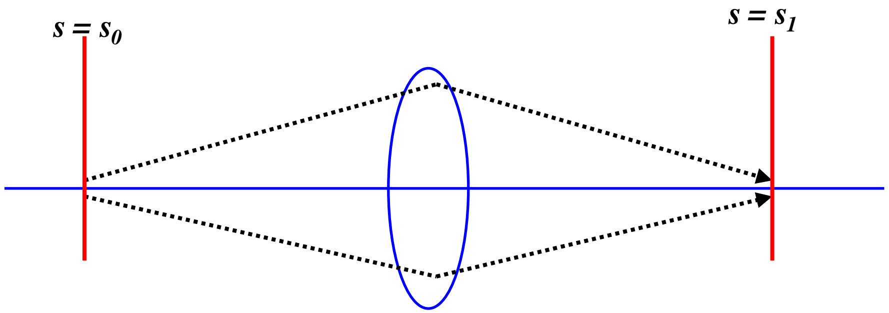
$$K > 0 \qquad x(s) = x_0 \cos(\sqrt{K}s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K}\sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s)$$

$$\sqrt{K} = \omega$$

Solution of piece-wise Hill's equations

What does this mean? Notice that the x' variable receives a negative kick, which corresponds to pointing the particle more towards the axis



Also note that the final coordinates are linear combinations of the initial coordinate, which is a direct consequence of linearising the equations of motion.

This means we can write the solution very compactly as a matrix equation

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M_{\text{quad}} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

A defocusing quadrupole

What about the case of K < 0? This has equation of motion

$$x'' - K \cdot x = 0$$

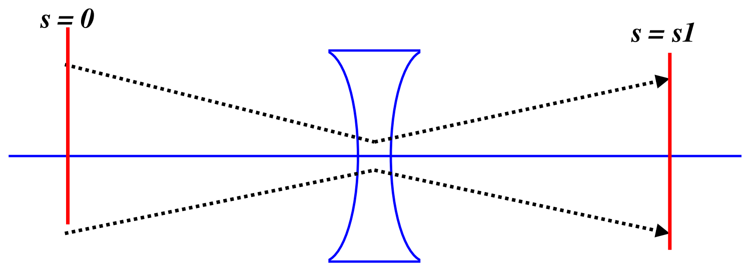
And corresponds to a diverging solution. We remember our studies of a mass on a spring and use the sinh/cosh functions

$$f(x) = \cosh(x) \quad f'(x) = \sinh(x)$$

Which means the general solution to the equation of motion is

$$x(s) = c_1 \cosh(\sqrt{|K|}s) + c_2 \sinh(\sqrt{|K|}s)$$

This corresponds to a defocusing lens, and we can write the matrix in the same kind of form but using sinh/cosh functions



$$M_{\mathrm{defoc\,quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

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Solution of piece-wise Hill's equations

$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$

$$z'' - kz = 0$$

These are our equations of motion, the horizontal plane and the vertical plane

This means for a given quadrupole magnet with no curvature, we have one sign of the constant K in one plane (say x) and the opposite sign in the other plane.

So a quadrupole, really because of Maxwell's equations and the need that the magnetic field is curl free, focuses in one plane and defocuses in another.

So by convention we call a horizontally focusing quadrupole a "focusing quadrupole" and vice versa.

$$\frac{e}{p}\frac{dB_z}{dx} = \frac{g}{B\rho} = k$$

Therefore a quadrupole with a positive sign for dB_z/dx gives $k > 0$, and so is horizontally focusing. Therefore $dB_z/dx < 0$ is horizontally defocusing.

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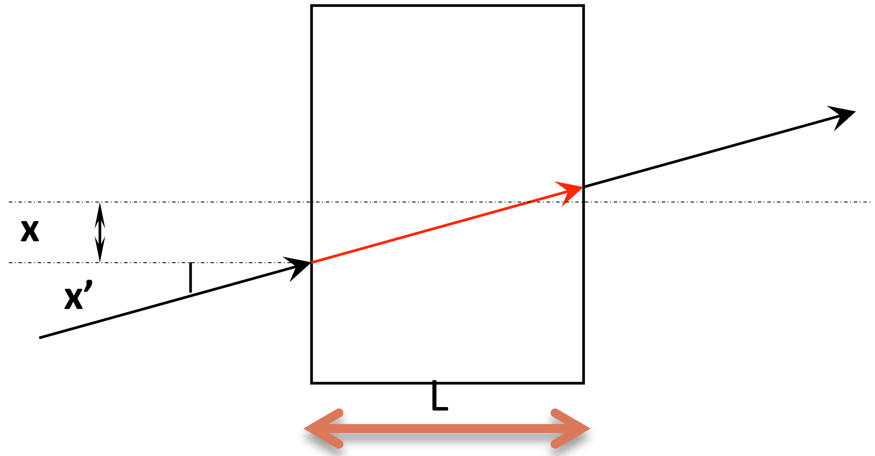
The drift space



A drift space is a region of the beam line with no electromagnetic fields. We can figure out the evolution equations for x and x' by either simple geometry or taking a limit of the quadrupole matrices for K->0. Either way we find the variables change as

$$x(s) = x_0 + x_0' \cdot L$$

$$x'(s) = x_0'$$

Which we can write as a matrix very easily

$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

The thin lens approximation

We know the matrix of a quadrupole can be written as

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

Very often the lens is short compared to the focal length

$$f = \frac{1}{Kl_q} \gg l_q$$

This means we can take the limit of very short magnet, whilst keeping the focal length constant:

$$|K| \to \infty \qquad l_q \to 0 \quad (Kl_q) = \text{const}$$

This give us the thin lens matrices, which are very useful for quick calculations of a given accelerator structure (recall K > 0 for a horizontally focusing quadrupole, so f > 0)

$$M_{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad \frac{1}{f} = Kl_q$$

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Now we know four linear transport matrices

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$M_{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

How would you turn these into a simple tracking code?

When would your code give valid results?

What's the determinant of these matrices?

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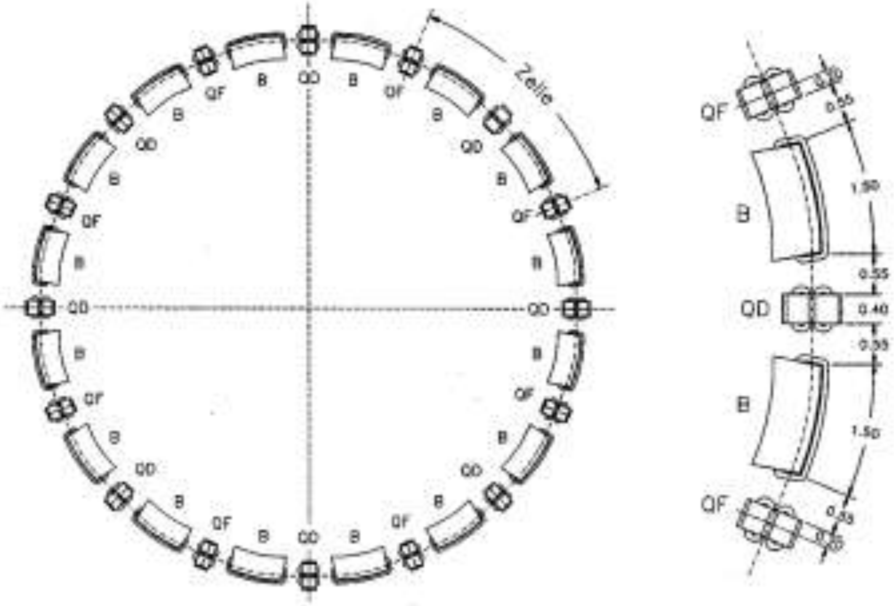
But we have lots of elements?



In a real accelerator we have lots of elements in a line.

Each element is represented by a matrix.

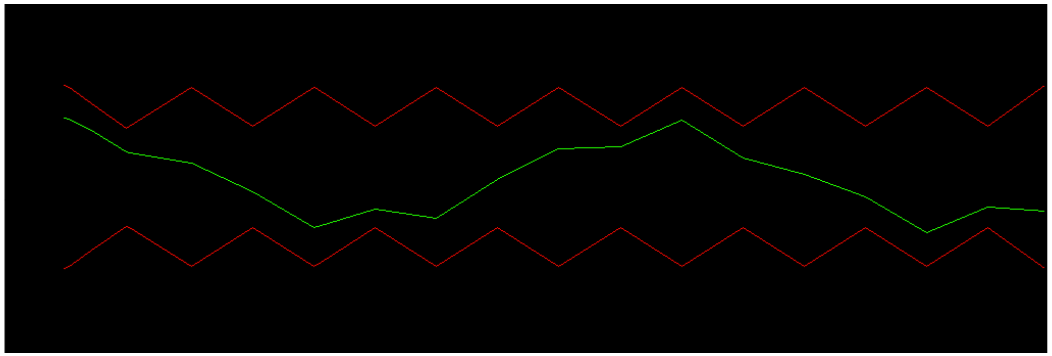
How do we transform through these sequences of elements?

Easy! We multiply the matrices of each element to give an overall transfer matrix through the system

$$M_{\mathrm{cell}} = M_{\mathrm{QF}} \cdot M_{\mathrm{bend}} \cdot M_{\mathrm{QD}} \cdot M_{\mathrm{drift}} \cdot M_{\mathrm{QF}}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s=1} = M(s=1, s=0) \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s=0}$$

First element!



Note, to compute the particle motion we need to know the fields and magnet positions very well!

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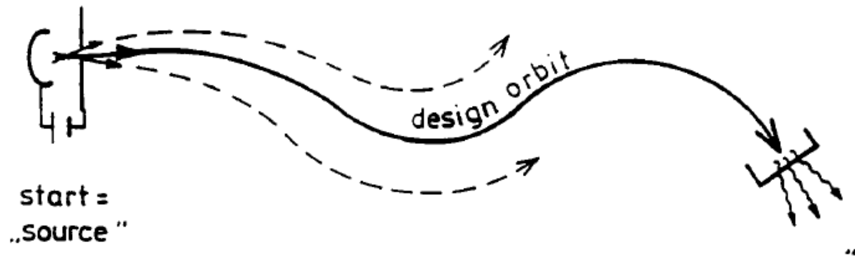
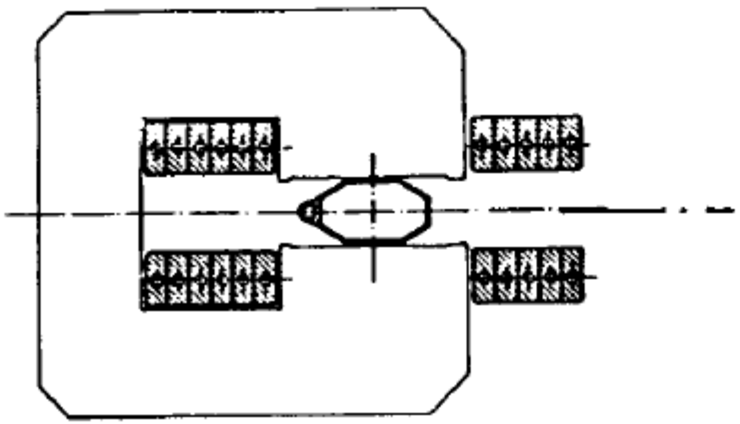
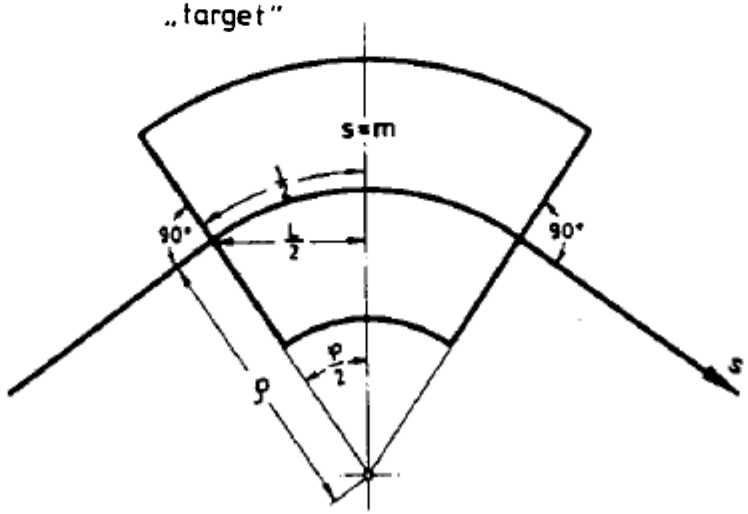
The dipole bend – assumption of a curved coordinate system

Is there a transfer matrix for a dipole? Yes, there is.

First of all, we used a curved coordinate system so the design trajectory follows
The curvature of the magnet







A pure dipole

To get the matrix, we start from the matrix of a quadrupole

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

In the horizontal plane, we have a pure bending contribution to K

$$K = \frac{1}{\rho^2}$$

And so we get for a pure bend of length l

$$M_{\text{dipole,x}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \qquad \theta = l/\rho$$

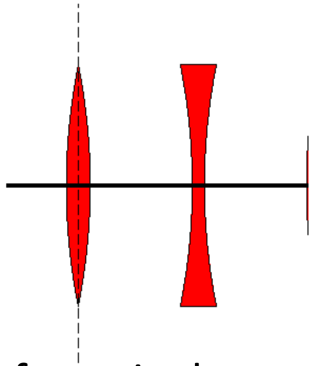
We can see that there is geometric focusing in the plane of the bend

The matrix in the vertical plane is a drift

What's the matrix for an angle of 2Pi? Is it sensible?

The doublet

Consider a doublet, which the the simplest example of a system with two quadrupoles



Focal lengths f_1 f_2
Separation d

The thin lens map for a single quadrupole is now known, as is the drift map
If we multiply these matrices, the focal length for the combined system is easily obtained from (2,1) element of the composite matrix

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

If I let $f_1 = - f_2$ then the leading terms cancel and the doublet is **focusing in both planes at the same time**. $f=f_1^2/d$ for x and y. This is a very pleasant feature!

This feature led to the invention of the alternating gradient (or strong focusing) principle and is the basic building block of modern accelerator lattices.

Can you see in terms of tracing rays through the two lenses why it happens?

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Maps

We have quietly brought in the idea of a map

The matrix M is that map that brings an initial state vector to a final state vector

$$X(s_1) = M(s_1|s_0)X(s_0)$$

For the linear case, the map can be represented as a matrix, so we have find a matrix representation of the map

Matrix representation ←→ Linear system

For non-linear systems, matrices do not work any more and we need to find new representations of the maps, for example Taylor maps or Lie maps.
This, however, is a little beyond this course.

Matrix algebra and transfer matrices

We know how to combine matrices, with a standard rule

$$M(s_2|s_0) = M(s_2|s_1)M(s_1|s_0)$$

Matrix algebra is not commutative

But it is associative and we can form matrix sub-groups
(Provided we maintain the order of the matrices!!)

One particularly useful map is the **one-turn map**

If we start at location s in a ring of circumference C, then the one turn map is defined as one turn around the ring

$$M(s+C|s)$$

This means the map for N revolutions of the ring is found from N applications to a given particle state vector of the one turn map

$$M(s+C|s)^N$$

More on this idea later…

Why are matrices useful?

We've basically re-written our equations in terms of matrices. Is this useful?

Yes, as it means we can use all the formal machinery of linear algebra e.g.

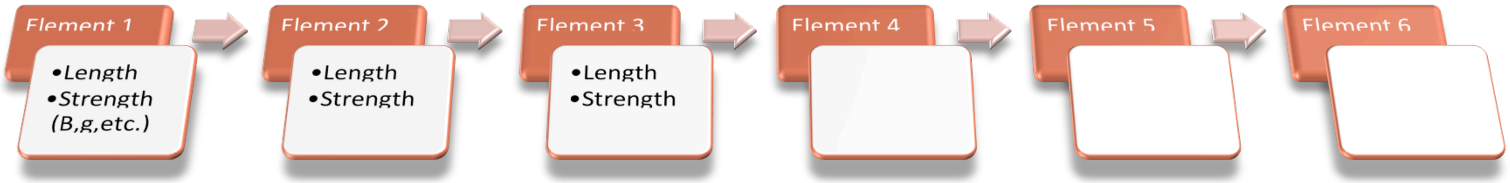
Matrix multiplication
eigenvalues
eigenvectors
Traces
Similarity transforms

Plus quickly see the impact on the beam of a series of elements.

All this is very powerful

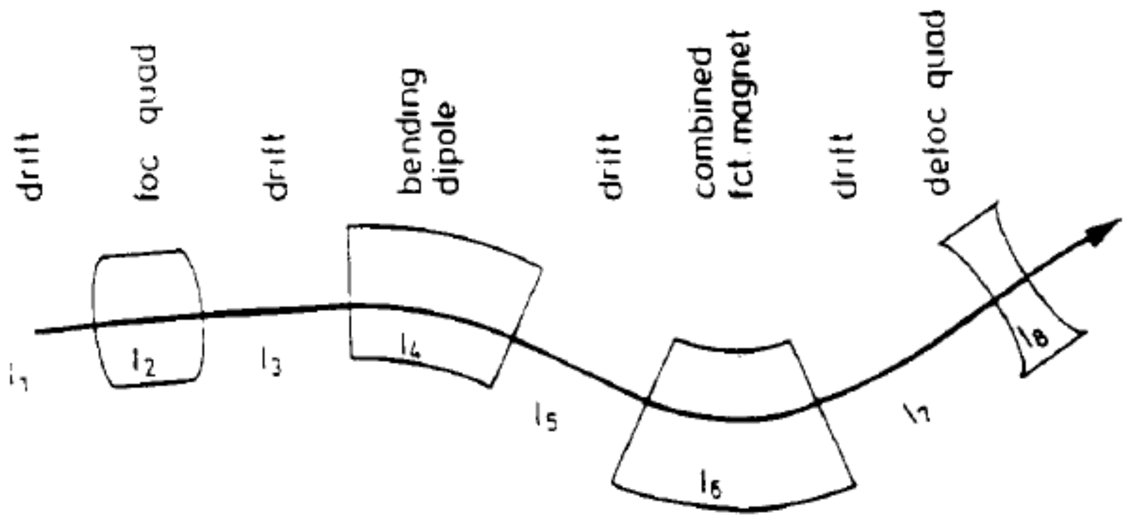
So how do we do it in a code?

Accelerator codes simply assume a piecewise-continuous representation of the accelerator structure.



$$R_T = R_n.R_{n-1}...R_3.R_2.R_1$$

but because of edge focusing the number of matrices is **not** the same as the number of elements.



Real lattices and lattice design

Remember that when we design lattices, that eventually it will get built (hopefully!)

Reality uses up more space than ideal elements….

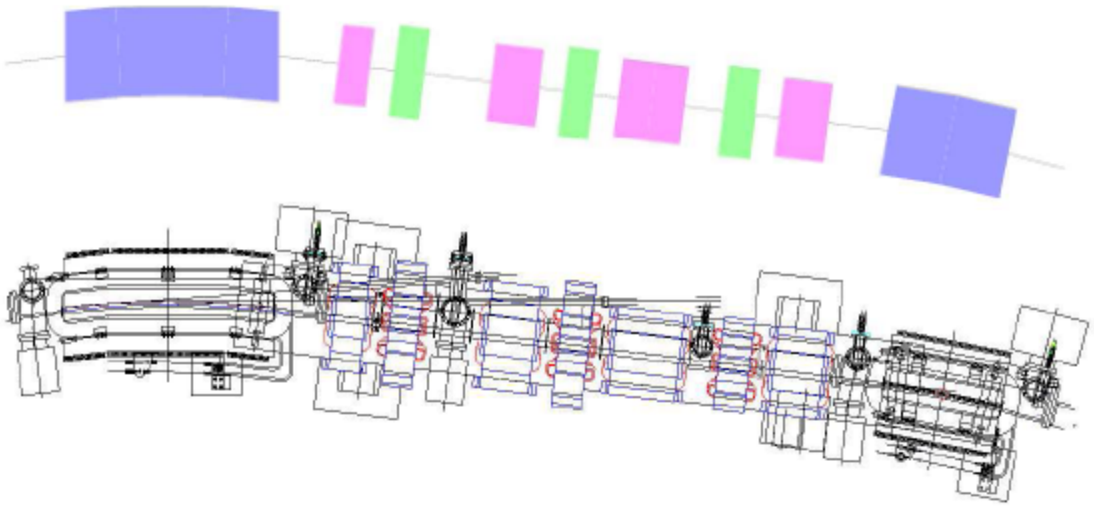


Fig. 1: Lattice sections as seen by the lattice designer (top) and the design engineer (bottom). Note how the space between ideal magnets is consumed by coils, beam-position monitors, absorbers, pumps, etc.

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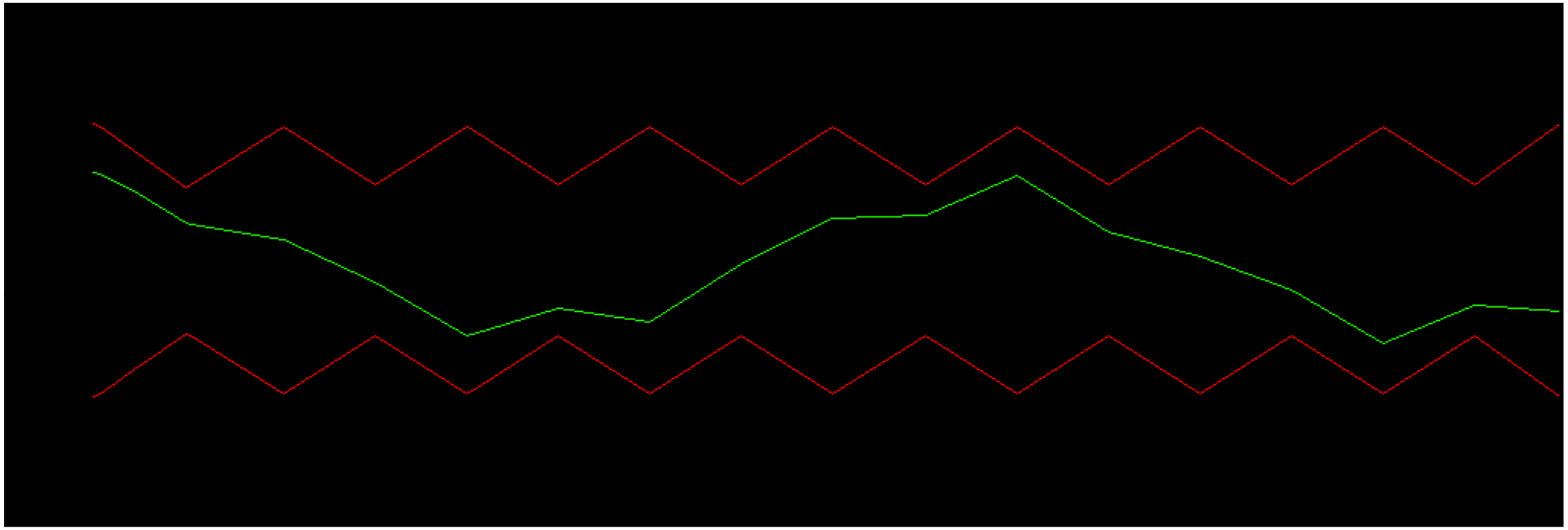
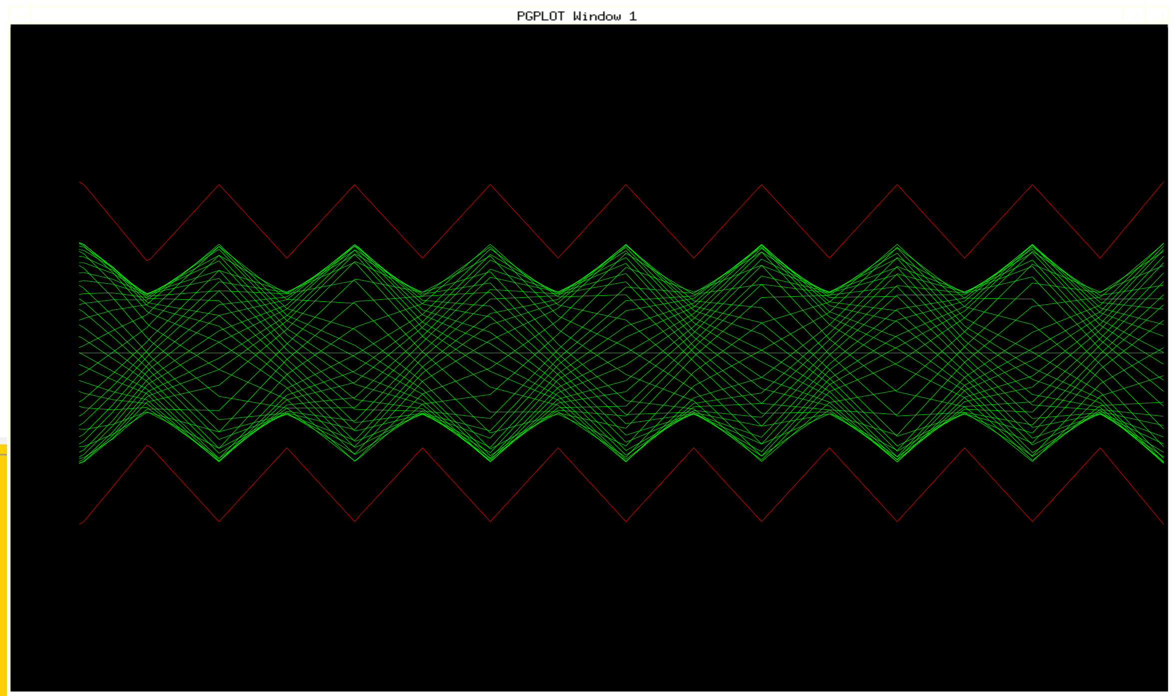
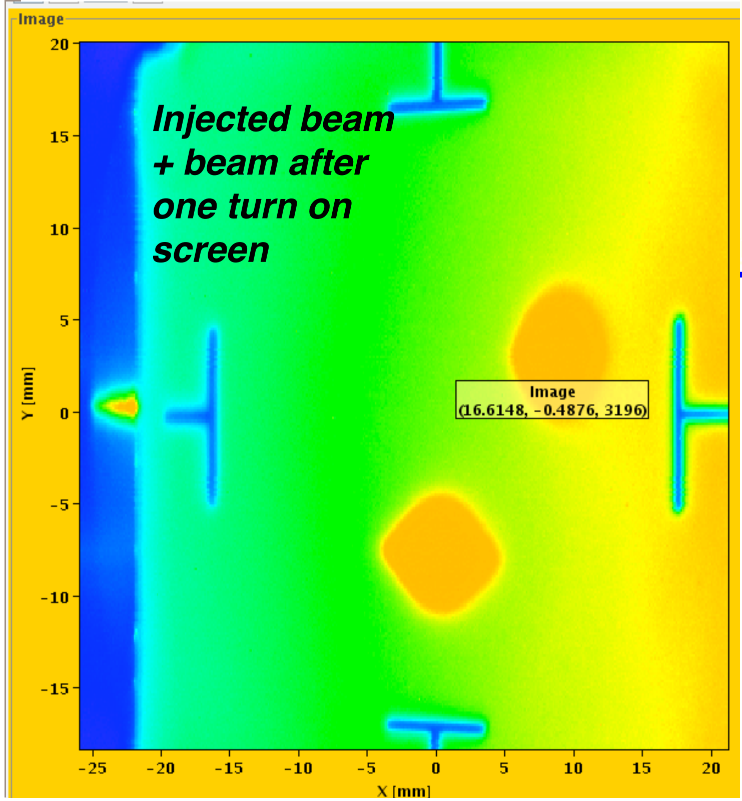
A single particle for a single turn

And, now, we have a particle in our accelerator for a few elements…



What if it wants to make many many turns?





So lots of particles over many turns form a kind of envelope of the beam.

Note we can see the straight reference particle in this plot, which is not a real particle.

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PART THREE- THE LATTICE FUNCTIONS

Back to Hill's equation

$$\boxed{x(s)'' + \left(k(s) + \frac{1}{\rho(s)^2}\right) x(s) = 0}$$

Hill's equation is a second order differential equation for a system with periodic focusing properties

It's like a pendulum but the restoring force is not a constant (like gravity is a constant on a pendulum)

In fact, the variable spring constant k(s) depends on the magnetic properties of the ring

If this ring has periodicity L, then so does the function k(s)

$$k(s+L) = k(s)$$

So we can expect a kind of quasi-harmonic oscillation, where the frequency and amplitude depend on the location in the ring and show periodicity similar to that of the function k(s)…

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The Courant-Snyder formalism

In the Courant-Snyder formalism we assume a solution of Hill's equation inspired by our intuition on position dependent amplitude and phase. It is this....

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$

- β(s) has the physical meaning of an amplitude which depends on the position around the accelerator.
- Ψ(s) is a position dependent phase
- ε is a constant. Because Hill's equation is linear, the constant does not appear in it. We'll see later that ε is special and is called the emittance

β(s) is the key quantity in the Courant-Synder formalism and has many names : the beta function, the beam envelope function, the Courant-Synder beta function, the amplitude function and so on. It is always positive.

We'll see that it represents the focusing properties of a lattice, and a small beta function means a tightly focused lattice, and vice versa.
The periodicity of the magnetic system is very important, and we will see

$$\beta(s + L) = \beta(s)$$

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A differential equation for the beta function

If we take the derivatives of the Courant-Synder ansatz and substitute into the equation of motion, we find we get two terms, one proportional to cosine and one proportional to sine. This is a good exercise to do! The coefficients of these terms must vanish separately, and we eventually obtain two differential equations

$$\frac{1}{2}(\beta\beta'' - \frac{1}{2}\beta'^2) - \beta^2\psi^2 + \beta^2 k = 0$$

$$\beta'\psi' + \beta\psi'' = 0$$

The second equation equation can be integrated immediately, since

$$\beta'\psi' + \beta\psi'' = (\beta\psi')'$$

And we are free to choose the integration constant to be unity

$$\beta\psi' = 1$$

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The Courant-Synder (or lattice) functions

We then immediately have the result for the phase function

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

So this position dependent phase is related to an integration of the beta function along the beam line, and knowing the beta function means we can compute the phase function

We can now eliminate the phase function from the first of the differential equations to get a differential equation for the beta function

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 k = 1$$

So knowing the distribution of focusing strengths along a beam line determines beta, although we never actually solve this equation in practice! Finally, we define the two functions (with beta, called the lattice functions)

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} \qquad \gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}$$

The appearance of the ellipse

Once the beta function is known, and hence alpha and gamma, the motion of a single particle is completely specified by specifying the emittance and the initial phase factor of the particle. So we have

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta}}\left[\alpha(s)\cos(\psi(s) + \psi_0) + \sin(\psi(s) + \psi_0)\right]$$

We can combine these two equations to give the quantity

$$\beta x' + \alpha x = -\sqrt{\epsilon\beta}\,\sin(\psi + \psi_0)$$

Kind of looks conjugate to x

Which means we can write an expression which is invariant for a particle

$$x^2 + (\beta x' + \alpha x)^2 = \epsilon\beta$$

Or, expanding the square, we arrive at the famous result

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

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The emittance is an invariant

Let's look at this equation carefully.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

For every point in the accelerator we have a value of the functions alpha, beta and gamma. They depend on the lattice.

At at particular point, if we combine the particle position and angle with these lattice functions we get an invariant, which was the emittance we saw in the solution to Hill's equations in the Courant-Synder formalism

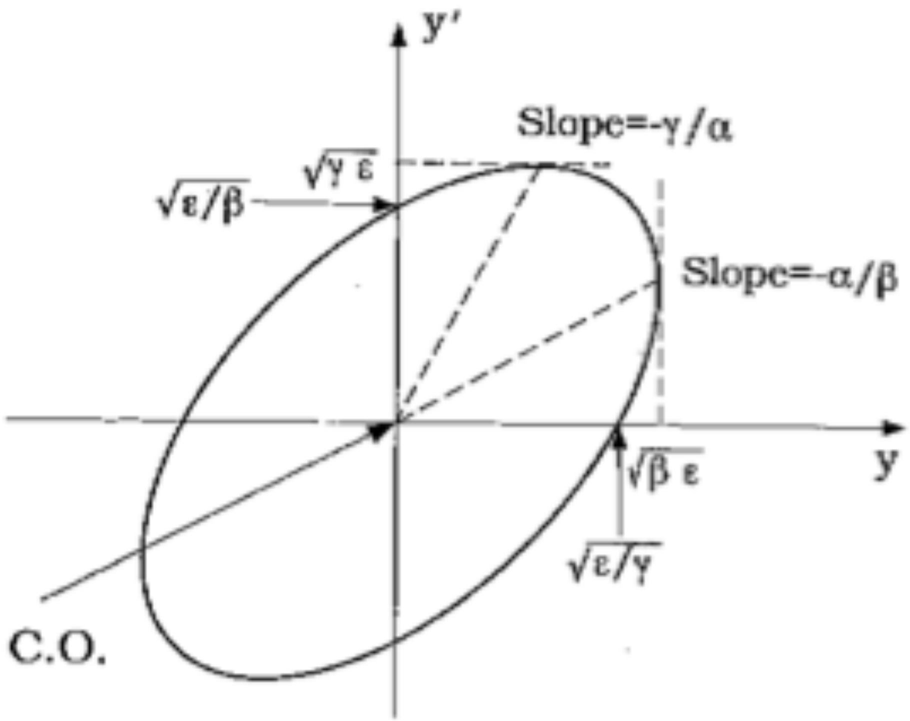
As the particle moves to the next location in the accelerator, where we have different lattice functions, it get a different position and angle. However if we form this function again at the new location we get the same value as before.

In other words, the emittance is a constant of the particle. (Although emittance is a bad name for it really……….a coffee discussion!)

The ellipse

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

This equation describes an ellipse in the (x,x') plane i.e. the ellipse boundary is the set of (x,x') points which obey this equation.



The area of the ellipse is given by

$$A = \pi\epsilon$$

The ellipse

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

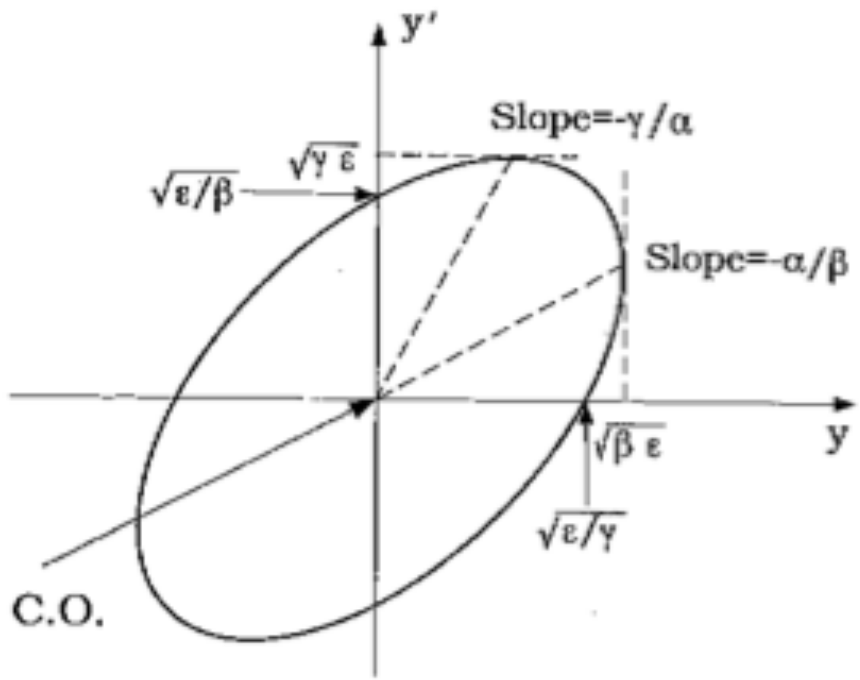
You may have seen this equation before in geometry.

If not, imagine there was no xx' term – what would it look like in the (x,x') plane?

This function actually describes an ellipse in the (x,x') plane, with ellipse parameters described by the values of alpha, beta and gamma.

Beta controls the extent along the x axis, gamma controls the extent along the x' axis, alpha tells you how upright the ellipse is (example – what values of alpha, beta and gamma give you a perfect circle?)

Exercise: If a particle sits on an ellipse, which way around the ellipse does it move?



The area of the ellipse is given by

$$A = \pi\epsilon$$

Which means the area of an ellipse transcribed by a given particle is **constant**

A stroboscopic plot of a particle turn after turn after turn

Recall that the lattice parameters are functions of the focusing of the lattice, so every point in the lattice has a value of the lattice functions and so every point in the lattice has it's own orientation of the ellipse.

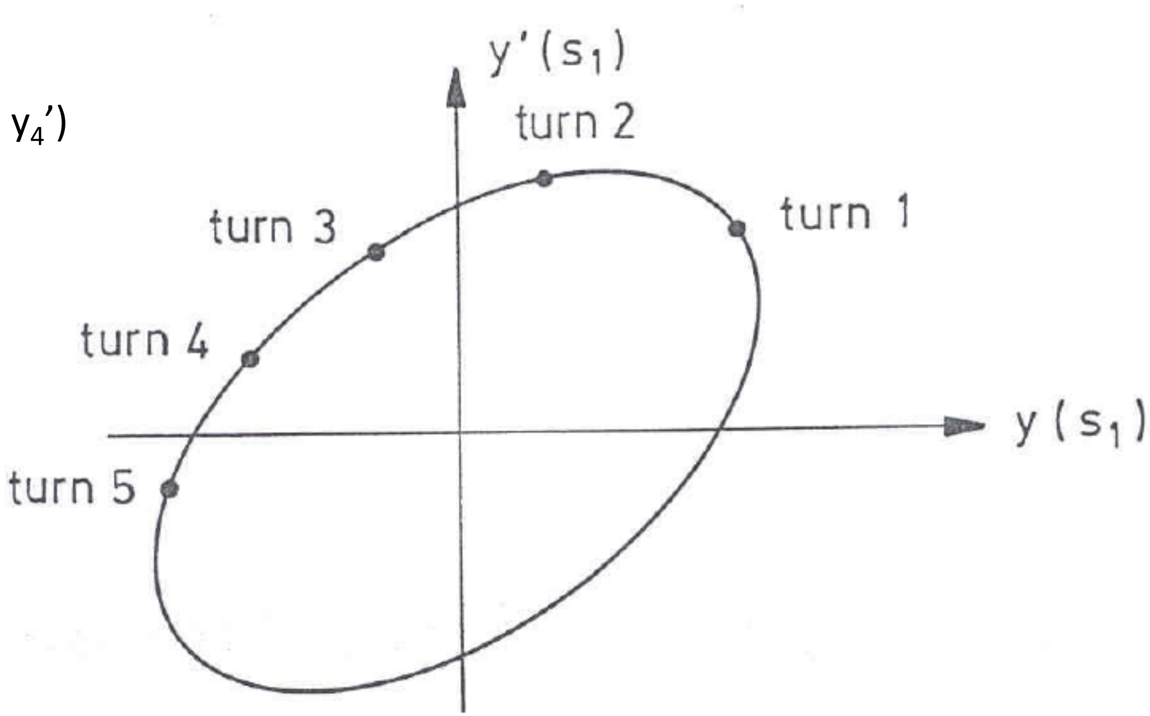
A given particle has it's own value of the emittance, so setting the area of the ellipse it moves around.

Let's play a game…we sit at a fixed point in the ring and watch a single particle turn after turn after turn. This can be done with a simple computer code.

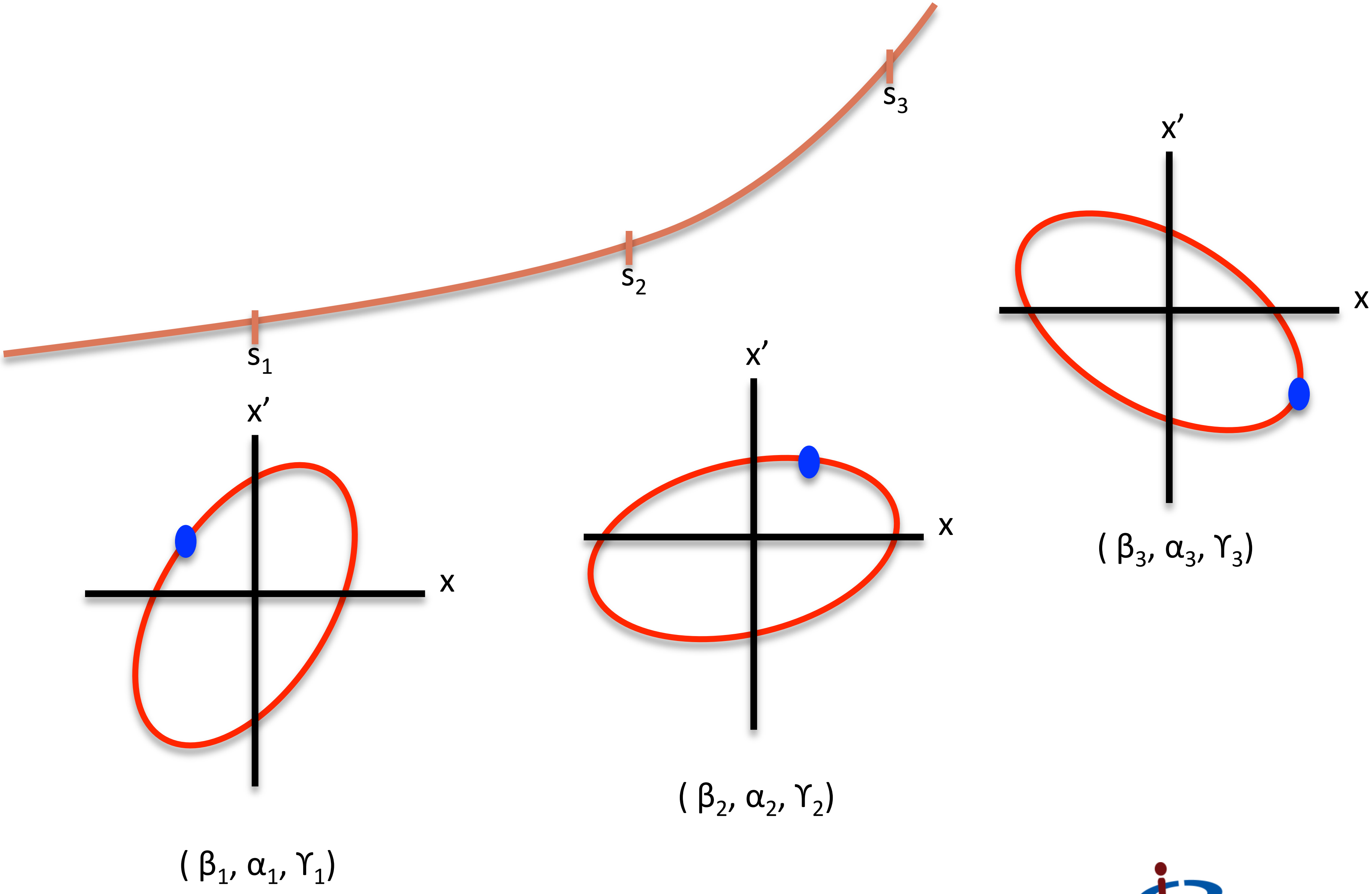
(y_1, y_1') (y_2, y_2') (y_3, y_3') (y_4, y_4')

All these points lie on the ellipse.

Note the particle jumps around the ellipse and does not move around it continuously



Moving along a beam line



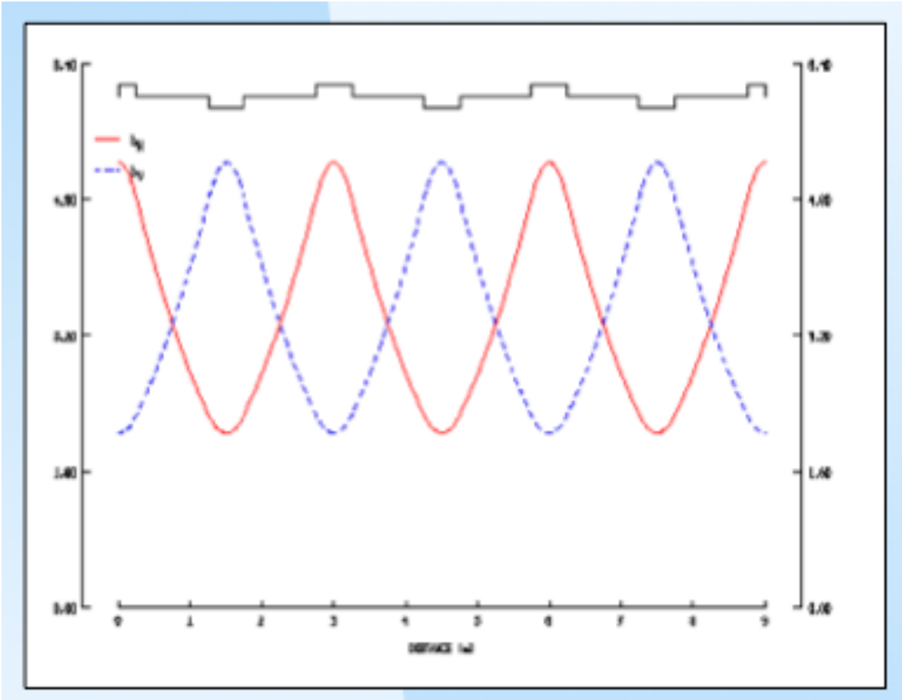
The beta function

The beta function is a central quantity in the Courant-Synder formalism

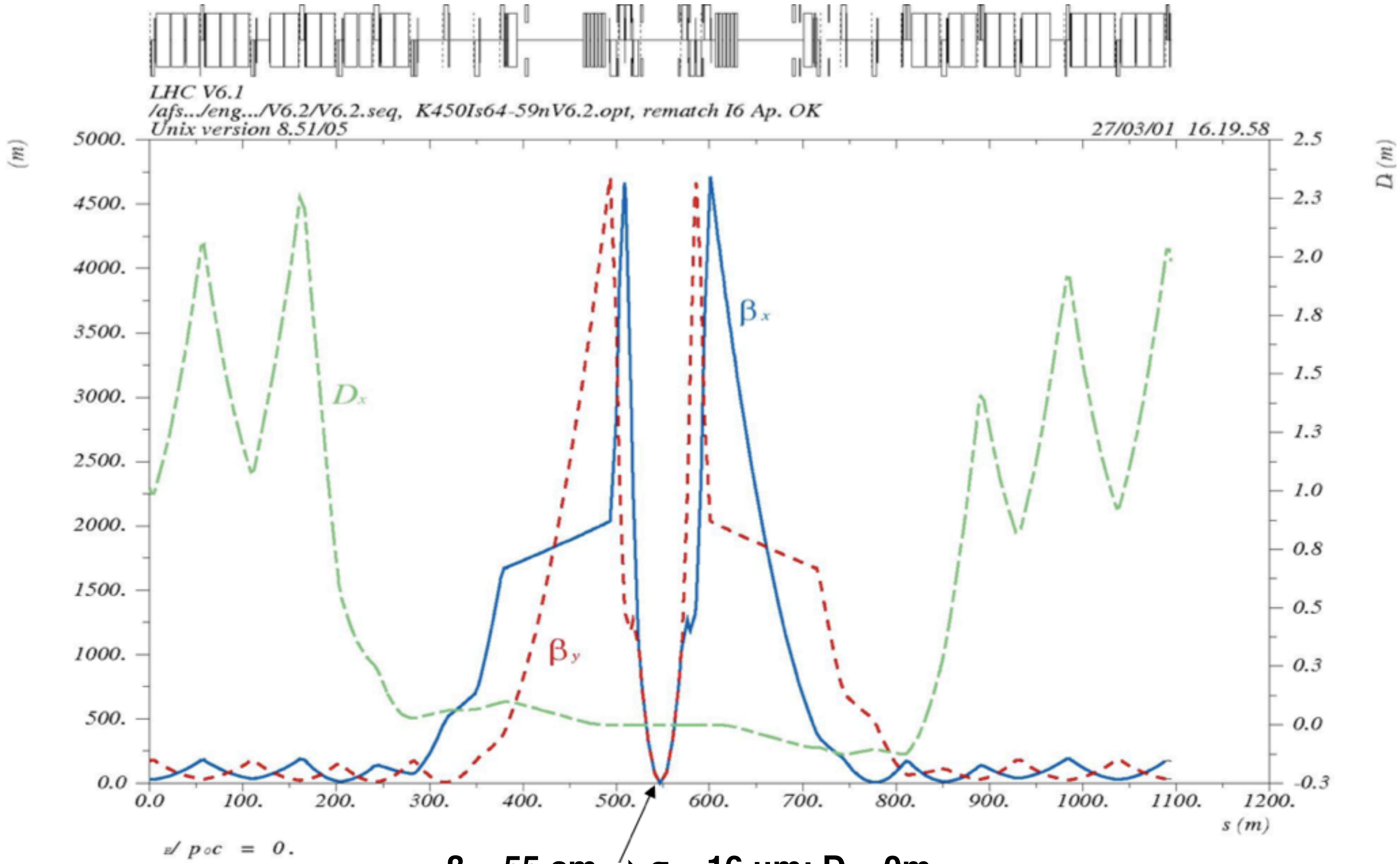
It is a positive function of position in the machine, and carries the same periodicity that the lattice itself carries.

It is determined by the focusing properties of the lattice, and is a function which is routinely computed in the design and operation of particle accelerators.
It is maximised in a focusing quadrupole and minimized in a defocusing quadrupole.

Let us now look at some examples.

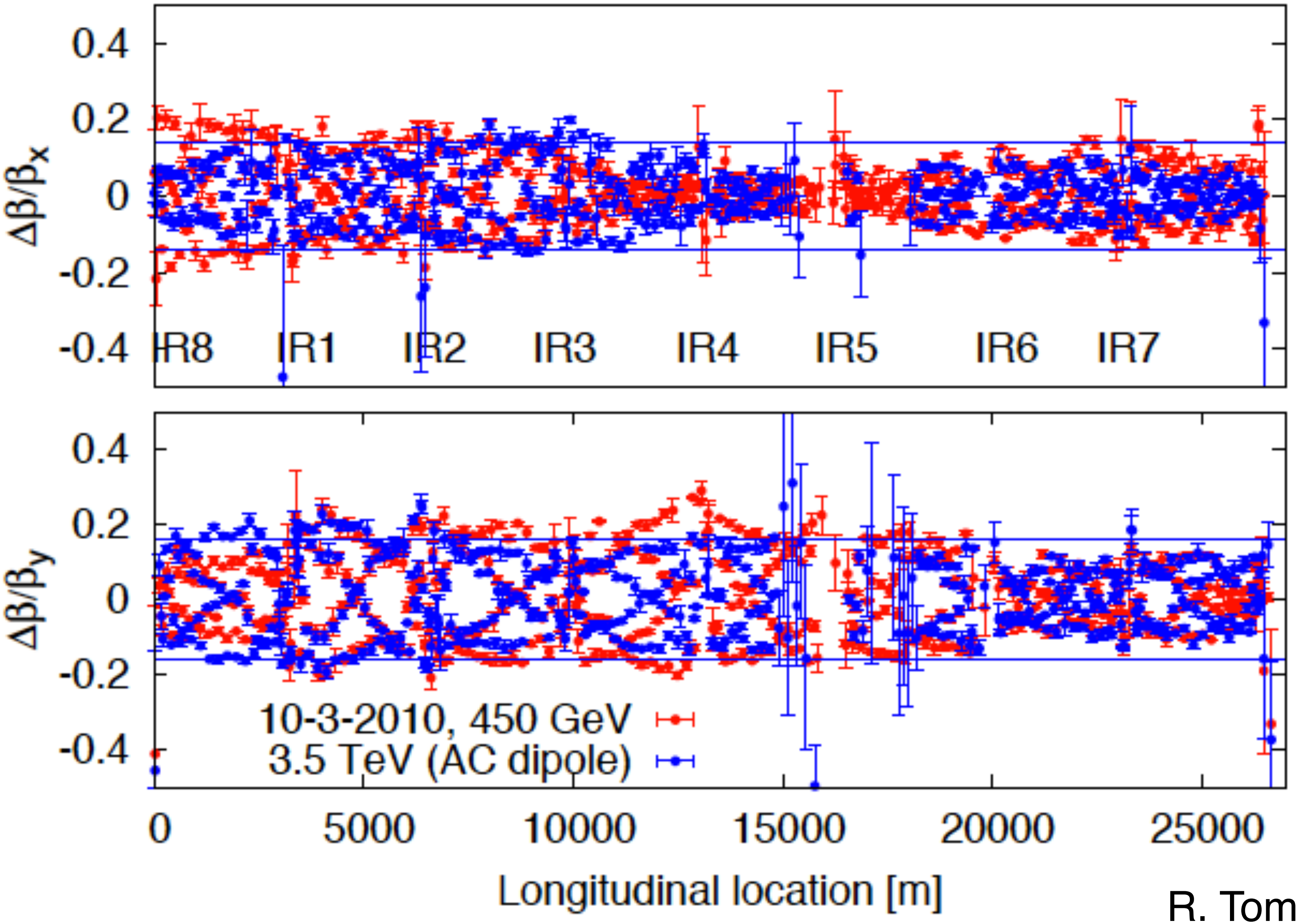


The beta function of the LHC



β = 55 cm → σ = 16 μm; D = 0m

And you can measure the beta function too!



R. Tomàs

Determination of the transfer matrix in terms of the Courant-Snyder functions

Let us now write a general transfer matrix between two points in terms of the lattice functions at these two points. To begin with, we return to the Courant-Synder form of the solution to Hill's equation, note it depends on two constants and write this ansatz in a slightly different form

$$x(s) = c_1\sqrt{\beta(s)}\cos\psi(s) + c_2\sqrt{\beta(s)}\sin\psi(s)$$

Where c_1 and c_2 are constants yet to be determined. If we define the initial conditions at the point '0' to be

$$\beta(0) = \beta_0 \quad \alpha(0) = \alpha_0 \quad \psi(0) = 0$$

And write the initial particle coordinates to be x_0 and x_0' then we can fix the unknown constants to be

$$c_1 = \frac{x_0}{\sqrt{\beta_0}} \qquad c_2 = \sqrt{\beta_0}x_0' + \frac{\alpha_0}{\sqrt{\beta_0}}x_0$$

And so we can write x(s) in the form

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0\sin\psi(s)\right]x_0 + \sqrt{\beta_0\beta(s)}x_0'\sin\psi(s)$$

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Determination of the transfer matrix in terms of the Courant-Snyder functions

We see the expression for x(s) is linear in x_0 and x_0'.

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0 \sin\psi(s)\right] x_0 + \sqrt{\beta_0\beta(s)}x_0' \sin\psi(s)$$

Taking the derivative of this expression, we can cast this equation into a convenient matrix form (as it's linear)
(I'm not going to write down the derivative…consider it an exercise)

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M(s_1|s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \qquad (0) \rightarrow (1)$$

where

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

$$\psi = \psi(s_1) - \psi(s_0)$$

The subscripts 0 and 1 refer to the beginning and end of the transfer map.
This means the transfer matrix between two points is purely determined by the lattice functions at each point and the phase advance between the points!!

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The one-turn map

The one turn (strictly one period) map is a very important quantity. Starting with

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0 - \alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

The map for one turn of the ring means we come back to the same position, and so

$$\beta_1 = \beta_0 = \beta \quad \alpha_1 = \alpha_0 = \alpha \quad \gamma_1 = \gamma_0 = \gamma$$

And so the one turn maps is

$$M(s+L|s) = \begin{pmatrix} \cos\Psi + \alpha\sin\Psi & \beta\sin\Psi \\ -\gamma\sin\Psi & \cos\Psi - \alpha\sin\Psi \end{pmatrix}$$

$$\gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

The phase advance for the one turn is given by

$$\psi_1 - \psi_0 = \Psi$$

The one-turn map

We can use this map to figure out the lattice functions. If we multiply all the matrices for all the elements in the ring together, we obtain the total matrix for one turn of the machine (again, strictly, one period)

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

We can get the one turn phase from the trace of this matrix, comparing it to the form we have for the one turn map in terms of lattice functions

$$\Psi = \arccos\left(\frac{m_{11} + m_{22}}{2}\right)$$

We can get the lattice function from the other matrix elements.

$$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi} \qquad \gamma = -\frac{m_{21}}{\sin\Psi}$$

Note for the phase advance to be real valued and hence **stable**, we need

$$|\mathrm{Tr}M| \leq 2$$

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An exercise

Using the expressions for the lattice functions in terms of the matrix elements, show that the condition det(M)=1 is the same as

$$\gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

by using the expression

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

(Understand this by seeing that a 2x2 matrix has 4 free parameters if real. If we require det M=1, then this drops to 3. If we parameterise the matrix with the three lattice functions and the phase advance, then the equation for gamma drops these four parameters to three.)

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The one-turn map at a different location

Imagine we know the one turn map at one location, say s. Is there a way to figure it out at another location, say s', provided we know the transfer matrix M for s to s'?

The answer is yes. They are related to each other by a similarity transform, and

$$M(s'+C|s') = M(s'|s)\cdot M(s+C|s)\cdot M^{-1}(s'|s)$$

I am going to state this without proof (which I avoid as much as possible!). Similarity transforms come from matrix theory and all manner of nice properties such as identical eigenvalues and traces before and after the transformation.
(We don't use them very much in an introductory course but you will later)

Let's be concrete and denote the matrix M (from s to s') by

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Let's use this to figure out how the lattice functions transform from place to place if we know the transfer matrix.

The transformation of the lattice functions

Starting with the similarity transform,

$$M(s'+C|s') = M(s'|s)\cdot M(s+C|s)\cdot M^{-1}(s'|s)$$

We can express the one turn maps in terms of the lattice functions at the locations s and s'

$$M(s+L|s) = \begin{pmatrix} \cos\Psi + \alpha\sin\Psi & \beta\sin\Psi \\ -\gamma\sin\Psi & \cos\Psi - \alpha\sin\Psi \end{pmatrix}$$

And do about 1 page of algebra to obtain the lattice functions at point s' (or 1) in terms of the lattice functions at point s (or 0) and the elements of the matrix M. The answer is

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} m_{11}m_{22}+m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

So knowing M, we can transform the lattice functions to any point in the beam line. Needless to say, this expression is important!!!

The phase advance and tune

Several times we have used the phase advance for one turn of a ring (period). It is

$$\Psi = \int_s^{s+L} \frac{ds}{\beta(s)}$$

And so given by an integral over the beta function.

We often call the phase advance for one turn of a ring the **tune**, and express it in units of 2*pi

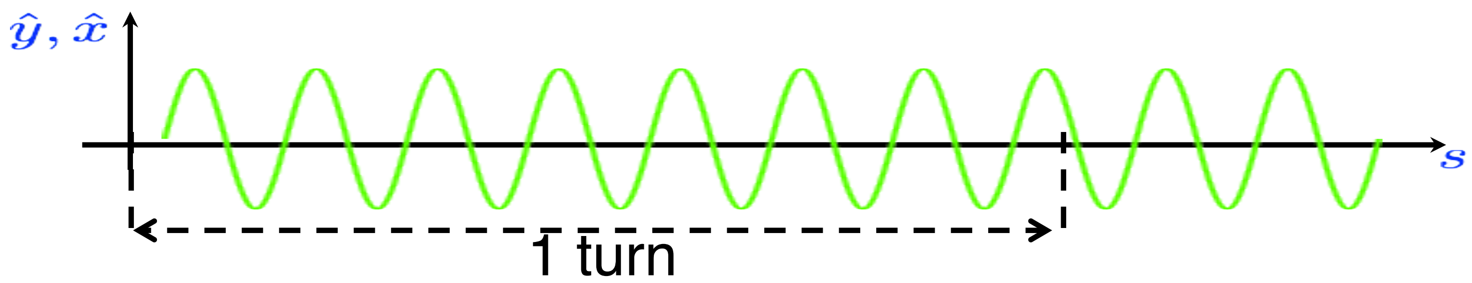
$$\nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$

(or Q)

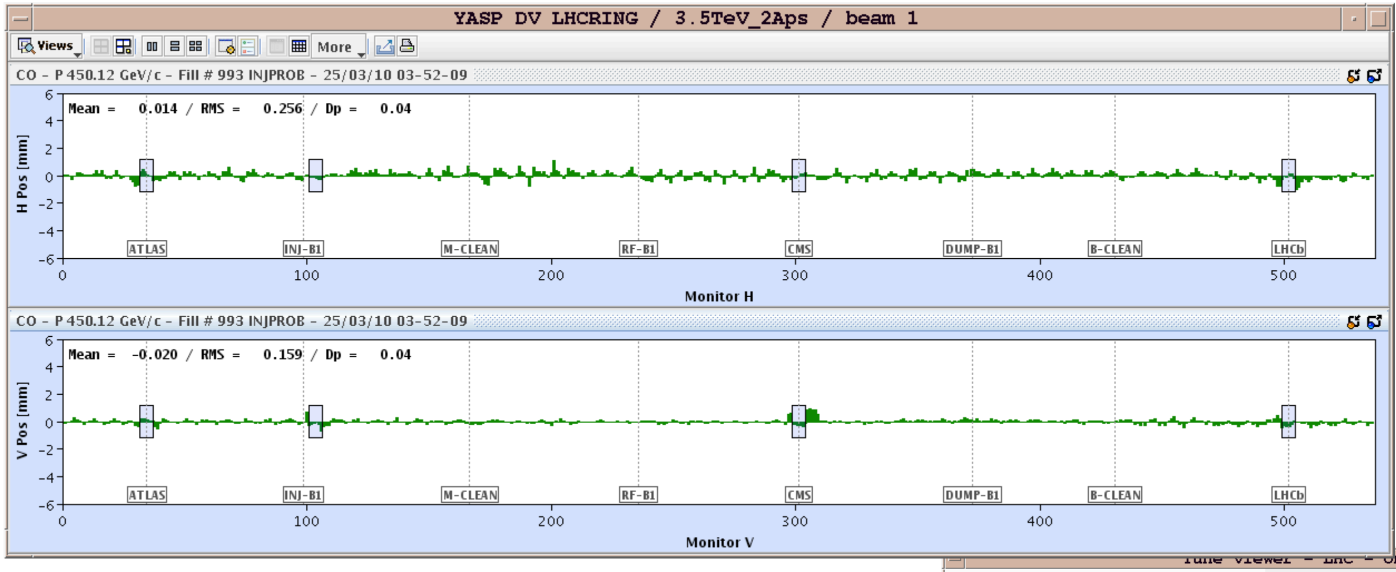
There is one tune for each plane, including the longitudinal plane, and it's a very important function for beam stability.
Note we can evaluate the tune at any point in the ring and always get the same answer (a property not shared by alpha, beta and gamma)
(This is because of the trace is invariant under similarity transformations)

Tune

$$\nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$



Tune

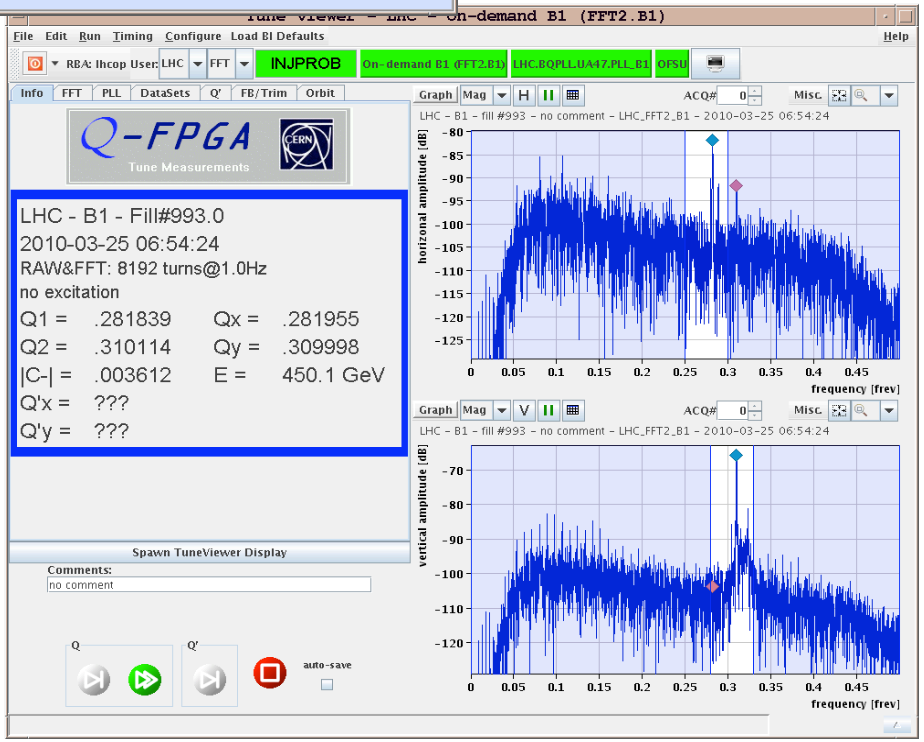


Tune is the number of oscillations made per turn

What is measurable and relevant is the non-integer part of the tune

.28

.31



Courant-Snyder parameter evolution in a drift

In a drift space of length L we have

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

And so

$$m_{11} = 1 \quad m_{12} = L \quad m_{21} = 0 \quad m_{22} = 1$$

The lattice functions evolve

$$\beta_1 = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha_1 = \alpha_0 - \gamma_0 L$$

$$\gamma_1 = \gamma_0$$

A particle evolves

$$x(L) = x_0 + L x'_0$$

$$x'(L) = x'_0$$

Is a drift stable?

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Exercise

Consider a thin lens quadrupole with focal length f.

Work out the change in the lattice functions through this quadrupole..

(Hint: what is the transfer matrix of the quadrupole and what expression tells you how the lattice functions evolve when you know the transfer matrix?)

Contrast the behaviour of a focusing ($f > 0$) and a defocusing ($f < 0$) quadrupole on the change in the beta function through the lens. (Hint: look at how alpha changes).

As a result, where would you expect to find the maximum horizontal beta function in a beam transport channel composed of alternating focusing and defocusing quadrupoles?